6.2 Dimensional Analysis

6.2.4 Common Dimensionless Parameters

- For a relationship between pressure drop Δp, characteristic length l, characteristic velocity V, density ρ, viscosity μ, gravity g, surface tension σ, speed of sound c, and angular frequency ω, i.e., Δp = *f*(l, V, ρ, μ, g, c, ω, σ)

- Using the π-theorem, with l, V, and ρ as repeating variables gives:

$$\frac{\Delta p}{\rho V^2} = f_1\left(\frac{V\rho l}{\mu}, \frac{V^2}{lg}, \frac{V}{c}, \frac{l\omega}{V}, \frac{V^2 \rho l}{\sigma}\right)$$

6.2 Dimensional Analysis

6.2.4 Common Dimensionless Parameters

- Each of the π-terms in the equation appears in many fluid flow situations.

$$\frac{\Delta p}{\rho V^2} = f_1\left(\frac{V\rho l}{\mu}, \frac{V^2}{lg}, \frac{V}{c}, \frac{l\omega}{V}, \frac{V^2\rho l}{\sigma}\right)$$

$$\text{Euler number, Eu} = \frac{\Delta p}{\rho V^2}$$

$$\text{Reynolds number, Re} = \frac{V\rho l}{\mu}$$

$$\text{Froude number}^2\text{, Fr} = \frac{V}{\sqrt{lg}}$$

$$\text{Mach number, M} = \frac{V}{c}$$

$$\text{Strouhal number}^2\text{, St} = \frac{l\omega}{V}$$

$$\text{Weber number}^2\text{, We} = \frac{V^2 l\rho}{\sigma}$$

6.2 Dimensional Analysis

6.2.4 Common Dimensionless Parameters

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$$\text{Weber number}^2,\ \text{We} = \frac{V^2 l\rho}{\sigma}$$

6.2 Dimensional Analysis

6.2.4 Common Dimensionless Parameters

- Each dimensionless number can be written as a ratio of two forces.

$$F_P = \text{pressure force} = \Delta p A \sim \Delta p l^2$$

$$F_I = \text{inertial force} = mV\frac{dV}{ds} \sim \rho l^3 V \frac{V}{l} = \rho l^2 V^2$$

$$F_\mu = \text{viscous force} = \tau A = \mu \frac{du}{dy} A \sim \mu \frac{V}{l} l^2 = \mu l V$$

$$F_g = \text{gravity force} = mg \sim \rho l^3 g$$

$$F_B = \text{compressibility force} = BA \sim \rho \frac{dp}{d\rho} l^2 = \rho c^2 l^2$$

$$F_\omega = \text{centrifugal force} = mr\omega^2 \sim \rho l^3 l \omega^2 = \rho l^4 \omega^2$$

$$F_\sigma = \text{surface tension force} = \sigma l$$

$$\text{Eu} \propto \frac{\text{pressure force}}{\text{inertial force}}$$

$$\text{Re} \propto \frac{\text{inertial force}}{\text{viscous force}}$$

$$\text{Fr} \propto \frac{\text{inertial force}}{\text{gravity force}}$$

$$\text{M} \propto \frac{\text{inertial force}}{\text{compressibility force}}$$

$$\text{St} \propto \frac{\text{centrifugal force}}{\text{inertial force}}$$

$$\text{We} \propto \frac{\text{inertial force}}{\text{surface tension force}}$$

6.2 Dimensional Analysis

6.2.4 Common Dimensionless Parameters

Table 6.2 Common Dimensionless Parameters in Fluid Mechanics

Parameter	*Expression*	*Flow situations where parameter is important*
Euler number	$\dfrac{\Delta p}{\rho V^2}$	Flows in which pressure drop is significant: most flow situations
Reynolds number	$\dfrac{\rho l V}{\mu}$	Flows that are influenced by viscous effects: internal flows, boundary layer flows
Froude number	$\dfrac{V}{\sqrt{lg}}$	Flows that are influenced by gravity: primarily free surface flows
Mach number	$\dfrac{V}{c}$	Compressibility is important in these flows, usually if $V > 0.3\,c$
Strouhal number	$\dfrac{l\omega}{V}$	Flow with an unsteady component that repeats itself periodically
Weber number	$\dfrac{V^2 l \rho}{\sigma}$	Surface tension influences the flow; flow with an interface may be such a flow

6.3 Similitude

6.3.1 General Information

- Study of predicting prototype conditions from model observations.
- If a model study has to be performed:
  - Need a quantity measured on the model (subscript *m*) to predict an associated quantity on the prototype (subscript *p*).
  - This needs **dynamic similarity between the model and prototype.**
  - Forces which act on corresponding masses in the model flow and prototype flow are in the same ratio throughout the entire flows.

6.3 Similitude

6.3.1 General Information

- If inertial forces, pressure forces, viscous forces, and gravity forces are present:

$$\frac{(F_I)_m}{(F_I)_p} = \frac{(F_P)_m}{(F_P)_p} = \frac{(F_\mu)_m}{(F_\mu)_p} = \frac{(F_g)_m}{(F_g)_p} = \text{const.}$$

Due to dynamic similarity at corresponding points in the flow fields.

Rearrange

$$\left(\frac{F_I}{F_P}\right)_m = \left(\frac{F_I}{F_P}\right)_p \qquad \left(\frac{F_I}{F_\mu}\right)_m = \left(\frac{F_I}{F_\mu}\right)_p \qquad \left(\frac{F_I}{F_g}\right)_m = \left(\frac{F_I}{F_g}\right)_p$$

$$\mathrm{Eu}_m = \mathrm{Eu}_p \qquad \mathrm{Re}_m = \mathrm{Re}_p \qquad \mathrm{Fr}_m = \mathrm{Fr}_p$$

6.3 Similitude

6.3.1 General Information (contd.)

- If only the forces above (previous slide) are present: $F_I = f(F_P, F_\mu, F_g)$

- Dimensional analysis lets the equation be written in terms of force ratios, as there is only one main dimension.

$$\mathrm{Eu} = f(\mathrm{Re}, \mathrm{Fr})$$

- If the Reynolds and Froude numbers are the same on the model and prototype, the Euler number should be the same.

- Guarantee dynamic similarity between model and prototype by equating the Reynolds number and Froude number of the model to that of the prototype.

6.3 Similitude

6.3.1 General Information (contd.)

- If compressibility forces are included, Mach number would be included.
- The inertial force ratio would be:

$$\frac{(F_I)_m}{(F_I)_p} = \frac{a_m m_m}{a_p m_p} = \text{const.}$$

If the mass ratio is a constant, then the acceleration ratio is a constant

Hence:

$$\frac{a_m}{a_p} = \frac{V_m^2/l_m}{V_p^2/l_p} = \text{const.}$$

6.3 Similitude

6.3.1 General Information (contd.)

- **Kinematic Similarity**: Velocity ratio is a constant between all corresponding points in the flow fields.
  - **Streamline pattern around the model is the same as that around the prototype except for a scale factor.**
- **Geometric Similarity**: Length ratio is a constant between all corresponding points in the flow fields.
  - **Model has the same shape as the prototype.**

6.3 Similitude

6.3.1 General Information (contd.)

For complete similarity between the model and prototype

- **Geometric similarity must be satisfied.**
- **Mass ratio of corresponding fluid elements is a constant.**
- **Dimensionless parameters (below) should be equal.**

$$\text{Euler number, Eu} = \frac{\Delta p}{\rho V^2}$$

$$\text{Reynolds number, Re} = \frac{V\rho l}{\mu}$$

$$\text{Froude number}^2\text{, Fr} = \frac{V}{\sqrt{lg}}$$

$$\text{Mach number, M} = \frac{V}{c}$$

$$\text{Strouhal number}^2\text{, St} = \frac{l\omega}{V}$$

$$\text{Weber number}^2\text{, We} = \frac{V^2 l\rho}{\sigma}$$

6.3 Similitude

6.3.1 General Information (contd.)

- Can now *predict* quantities of interest on a prototype from measurements on a model.

Drag forces, F_D	$\frac{(F_D)_m}{(F_D)_p} = \frac{(F_I)_m}{(F_I)_p} = \frac{\rho_m V_m^2 l_m^2}{\rho_p V_p^2 l_p^2}$	Equate ratio of drag forces to ratio of inertial forces.
Power input, $\dot{W}$	$\frac{\dot{W}_m}{\dot{W}_p} = \frac{(F_I)_m V_m}{(F_I)_p V_p} = \frac{\rho_m V_m^2 l_m^2 V_m}{\rho_p V_p^2 l_p^2 V_p}$	Power is force times velocity.

- Can predict a prototype quantity if we select the model fluid, the scale fluid, and the dimensionless number.

6.3 Similitude

6.3.2 Confined Flows

- A confined flow is a flow that has no free surface (liquid-gas surface) or interface (two different liquids).

- Can only move within a specific region (external flows around objects, or internal flows in pipes).

- Isn't influenced by gravity or surface tension.
- Dominant effect is viscosity in incompressible confined flows.
- Relevant flows are pressure, inertial, and viscous forces.
  - Dynamic similarity is obtained if the ratios between the model and the prototype are the same.
- **Hence, only the Reynolds number is the dominant dimensionless parameter.**
  - **npressibility effects are significant, Mach number would become important.**

6.3 Similitude

A test is to be performed on a proposed design for a large pump that is to deliver 1.5 m³/s of water from a 400-mm-diameter impeller with a pressure rise of 400 kPa. A model with an 80-mm-diameter impeller is to be used. What flow rate should be used and what pressure rise is to be expected? The model fluid is water at the same temperature as the water in the prototype.

Solution

For similarity to exist in this confined incompressible flow problem, the Reynolds numbers must be equal; that is,

$$\mathrm{Re}_m = \mathrm{Re}_p$$
$$\frac{V_m d_m}{\nu_m} = \frac{V_p d_p}{\nu_p}$$

Recognizing that $\nu_m = \nu_p$ if the temperatures are equal, we see that

$$\frac{V_m}{V_p} = \frac{d_p}{d_m} = \frac{0.4}{0.08} = 5$$

The ratio of flow rates is found recognizing that $Q = VA$:

$$\frac{Q_m}{Q_p} = \frac{V_m d_m^2}{V_p d_p^2} = 5 \times \left(\frac{1}{5}\right)^2 = \frac{1}{5}$$

Thus we find that

$$Q_m = \frac{Q_p}{5} = \frac{1.5}{5} = \underline{0.3\ \mathrm{m^3/s}}$$

The dimensionless pressure rise is found using the Euler number:

$$\left(\frac{\Delta p}{\rho V^2}\right)_m = \left(\frac{\Delta p}{\rho V^2}\right)_p$$

Hence the pressure rise for the model is

$$\Delta p_m = \Delta p_p \frac{\rho_m}{\rho_p}\frac{V_m^2}{V_p^2} = 400 \times 1 \times 5^2 = \underline{10\,000\ \mathrm{kPa}}$$

Note that in this example we see that the velocity in the model is equal to the velocity in the prototype multiplied by the length ratio, and the pressure rise in the model is equal to the pressure rise in the prototype multiplied by the length ratio squared. If the length ratio were very large, it is obvious that to maintain Reynolds number equivalence would be quite difficult, indeed. This observation is discussed in more detail in Section 6.3.4.

6.3 Similitude

6.3.3 Free-Surface Flows

- A free-surface flow is a flow where part of the boundary involves a pressure boundary condition.
    - E.g., Flows in channels, flows with two fluids separated by an interface, etc.

- Location and velocity of the free surface are unknown.
- Pressure is the same on either side of the interface (unless there is significant surface tension).
- Gravity controls the location and motion of the free surface.
- Viscous effects are significant

- **Requires the Froude number.**

6.3 Similitude

A 1:20 scale model of a surface vessel is used to test the influence of a proposed design on the wave drag. A wave drag of 27.6 N is measured at a model speed of 2.44 m/s. What speed does this correspond to on the prototype, and what wave drag is predicted for the prototype? Neglect viscous effects, and assume the same fluid for model and prototype.

Solution

The Froude number must be equated for both model and prototype. Thus

$$\mathrm{Fr}_m = \mathrm{Fr}_p \qquad \frac{V_m}{\sqrt{l_m g}} = \frac{V_p}{\sqrt{l_p g}}$$

This yields, recognizing that g does not vary significantly on the surface of the earth,

$$V_p = V_m\left(\frac{l_p}{l_m}\right)^{1/2} = 2.44\sqrt{20} = \underline{10.9\ \text{m/s}}$$

To find the wave drag on the prototype, we equate the drag ratio to the inertia force ratio:

$$\frac{(F_D)_m}{(F_D)_p} = \frac{\rho_m V_m^2 l_m^2}{\rho_p V_p^2 l_p^2}$$

This allows us to calculate the wave drag on the prototype as, using $\rho_p = \rho_m$,

$$(F_D)_p = (F_D)_m \frac{\rho_p V_p^2 l_p^2}{\rho_m V_m^2 l_m^2}$$

$$= 27.6 \times \frac{10.9^2}{2.44^2} \times 20^2 = \underline{220\ \text{kN}}$$

Note: We could have used the gravity force ratio rather than the inertial force ratio, but we could not have used the viscous force ratio since viscous forces were assumed negligible.

6.3 Similitude

A 1:10 scale model of an automobile is used to measure the drag on a proposed design. It is to simulate a prototype speed of 90 km/h. What speed should be used in the wind tunnel if Reynolds numbers are equated? For this condition, what is the ratio of drag forces?

Solution

The same fluid exists on model and prototype; thus, equating the Reynolds numbers results in

$$\frac{V_m l_m}{\nu_m} = \frac{V_p l_p}{\nu_p} \qquad \therefore\ V_m = V_p \frac{l_p}{l_m}$$

$$= 90 \times 10 = \underline{900 \text{ km/h}}$$

This speed would, of course, introduce compressibility effects, effects that do not exist in the prototype. Hence the proposed model study would be inappropriate.

If we did use this velocity in the model, the drag force ratio would be

$$\frac{(F_D)_p}{(F_D)_m} = \frac{\rho_p V_p^2 l_p^2}{\rho_m V_m^2 l_m^2} \qquad \therefore\ \frac{(F_D)_p}{(F_D)_m} = 1$$

Thus we see that the drag force on the model is the same as the drag force on the prototype if the same fluids are used when we equate Reynolds numbers.

6.3 Similitude

In Example 6.5, if the Reynolds numbers were equated, the velocity in the model study was observed to be in the compressible flow regime (i.e., $M > 0.3$ or $V_m > 360$ km/h). To conduct an acceptable model study, could we use a velocity of 90 km/h on a model with a characteristic length of 10 cm? Assume that the drag coefficient ($C_D = F_D / \frac{1}{2}\rho V^2 A$, where A is the projected area) is independent of Re for $\mathrm{Re} > 10^5$. If so, what drag force on the prototype would correspond to a drag force of 1.2 N measured on the model?

Solution

The proposed model study in a wind tunnel is to be conducted with $V_m = 90$ km/h and $l_m = 0.1$ m. Using $\nu = 1.6 \times 10^{-5}\ \mathrm{m^2/s}$, the Reynolds number is

$$\mathrm{Re}_m = \frac{V_m l_m}{\nu_m} = \frac{(90 \times 1000/3600) \times 0.1}{1.6 \times 10^{-5}} = 1.56 \times 10^5$$

This Reynolds number is greater than 10^5, so we will assume that similarity exists between model and prototype. The velocity of 90 km/h is sufficiently high.

The drag force on the prototype traveling at 90 km/h corresponding to 1.2 N on the model is found from

$$\frac{(F_D)_p}{(F_D)_m} = \frac{\rho_p V_p^2 l_p^2}{\rho_m V_m^2 l_m^2} \qquad \therefore (F_D)_p = (F_D)_m \cancelto{1}{\frac{\rho_p}{\rho_m}} \cancelto{1}{\frac{V_p^2}{V_m^2}} \frac{l_p^2}{l_m^2} = 1.2 \times 10^2 = \underline{120\ \mathrm{N}}$$

Note that in this example we have assumed that the drag coefficient is independent of Re for $\mathrm{Re} > 10^5$. If the drag coefficient continued to vary above $\mathrm{Re} = 10^5$ (this would be evident from experimental data), the foregoing analysis would have to be modified accordingly.

6.3 Similitude

6.3.5 Compressible Flows

- For most compressible flows, the Reynolds number is very large (not significant).
  - Mach number is the primary dimensionless parameter for model studies.

$$\mathrm{M}_m = \mathrm{M}_p \qquad \text{or} \qquad \frac{V_m}{c_m} = \frac{V_p}{c_p}$$

- If the study is carried in a wind tunnel (with a prototype fluid of air), $c_m = c_p$.
  - Assume the temperature is the same in both flows.

- In this case, the velocity in the model study is equal to the velocity associated with the prototype.

6.3 Similitude

The pressure rise from free stream to the nose of a fusilage section of an aircraft is measured in a wind tunnel at 20°C to be 34 kPa with a wind-tunnel airspeed of 900 km/h. If the test is to simulate flight at an elevation of 12 km, what is the prototype velocity and the expected nose pressure rise?

Solution

To find the prototype velocity corresponding to a wind-tunnel airspeed of 900 km/h, we equate the Mach numbers

$$M_m = M_p \qquad \text{or} \qquad \frac{V_m}{\sqrt{kRT_m}} = \frac{V_p}{\sqrt{kRT_p}}$$

Thus

$$V_p = V_m\left(\frac{kRT_p}{kRT_m}\right)^{1/2} = 900\left(\frac{216.7}{293}\right)^{1/2} = \underline{774\ \text{km/h}}$$

The pressure at the nose of the prototype fusilage is found using the Euler number as follows:

$$\frac{\Delta p_m}{\rho_m V_m^2} = \frac{\Delta p_p}{\rho_p V_p^2}$$

$$\therefore \Delta p_p = \Delta p_m \frac{\rho_p}{\rho_m}\frac{V_p^2}{V_m^2}$$

$$= 34 \times \frac{0.3119}{1.225} \times \frac{774^2}{900^2} = \underline{6.4\ \text{kPa}}$$

lensities and temperature T_p were found in Appendix B.

6.3 Similitude

6.3.6 Periodic Flows

- There are regions of flows in which periodic motions occur.
  - E.g., When fluid flows past a cylindrical object.
- **For these flows, we need to equate Strouhal numbers to model the periodic motion.**

$$\frac{V_m}{\omega_m l_m} = \frac{V_p}{\omega_p l_p}$$

- Additional dimensionless parameters that may be equated.
  - **In viscous flows → Reynolds number**
  - **In free-surface flows → Froude number**
  - **In compressible flows → Mach number**

6.3 Similitude

A large wind turbine, designed to operate at 50 km/h, is to be tested in a laboratory by constructing a 1:15 scale model. What airspeed should be used in the wind tunnel, what angular velocity should be used to simulate a prototype angular speed of 5 rpm, and what power output is expected from the model if the prototype output is designed to be 500 kW?

Solution

The speed in the wind tunnel can be any speed above that needed to provide a sufficiently large Reynolds number. Let us select the same speed with which the prototype is to operate, namely, 50 km/h, and calculate the minimum characteristic length that a Reynolds number of 10^5 would demand; this gives

$$\mathrm{Re} = \frac{Vl}{\nu} \qquad 10^5 = \frac{(50 \times 1000/3600) \times l}{1.6 \times 10^{-5}} \qquad \therefore l = 0.12 \text{ m}$$

Obviously, in a reasonably large wind tunnel we can maintain a characteristic length (e.g., the blade length) that large.

The angular velocity is found by equating the Strouhal numbers. There results

$$\frac{V_m}{\omega_m l_m} = \frac{V_p}{\omega_p l_p} \qquad \therefore \omega_m = \omega_p \cancelto{1}{\frac{V_m}{V_p}} \frac{l_p}{l_m} = 5 \times 1 \times 15 = \underline{75 \text{ rpm}}$$

assuming that the wind velocities are equal.

The power is found by observing that power is force times velocity:

$$\frac{\dot{W}_m}{\dot{W}_p} = \frac{\rho_m V_m^3 l_m^2}{\rho_p V_p^3 l_p^2}$$

or

$$\dot{W}_m = \dot{W}_p \cancelto{1}{\frac{\rho_m}{\rho_p}} \cancelto{1}{\frac{V_m^3}{V_p^3}} \frac{l_m^2}{l_p^2} = 500 \times \left(\frac{1}{15}\right)^2 = \underline{2.22 \text{ kW}}$$