THE DIFFERENTIAL FORMS OF THE FUNDAMENTAL LAWS

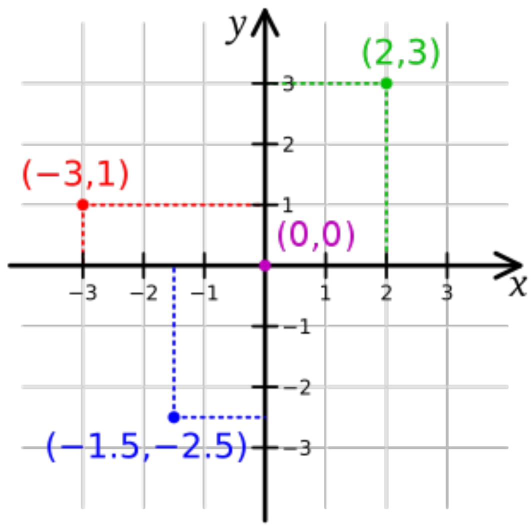
Coordinate system

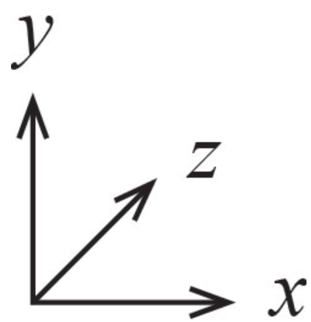
There are two common coordinate system:

1. Cartesian coordinate
2. Polar coordinate

CARTESIAN COORDINATE SYSTEM:

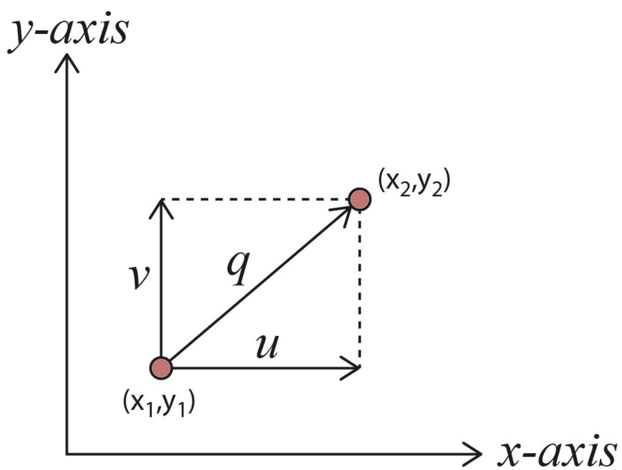
A Cartesian coordinate system is a coordinate system that specifies each point uniquely in a plane by a pair of numerical coordinates, which are the signed distances to the point from two fixed perpendicular directed lines, measured in the same unit of length. Each reference line is called a coordinate axis or just axis (plural axes) of the system, and the point where they meet is its origin, at ordered pair (0, 0). The coordinates can also be defined as the positions of the perpendicular projections of the point onto the two axes, expressed as signed distances from the origin.





Example of Cartesian coordinate

Motion of one element can be shown as below:



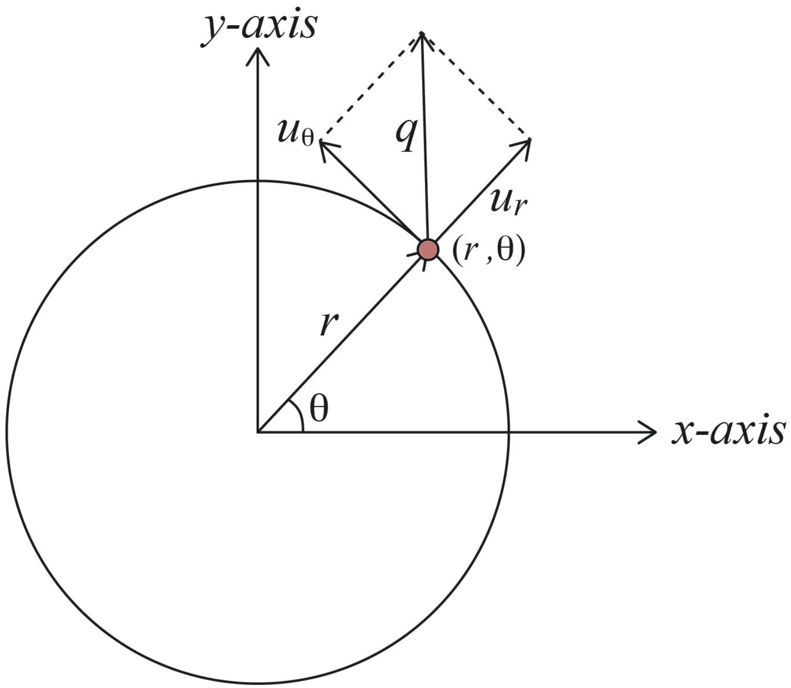
u = velocity at x-axis
v = velocity at y-axis
q = resultant velocity

$$q = \sqrt{u^2 + v^2}$$

POLAR COORDINATE SYSTEM:

In mathematics, the polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction.

The reference point (analogous to the origin of a Cartesian coordinate system) is called the pole, and the ray from the pole in the reference direction is the polar axis. The distance from the pole is called the radial coordinate or radius, and the angle is called the angular coordinate, polar angle, or azimuth.



$$u_r = \text{velocity at radial direction}$$
$$u_\theta = \text{velocity at tangential direction}$$
$$q = \text{resultant velocity}$$

$$q = \sqrt{(u_r)^2 + (u_\theta)^2}$$

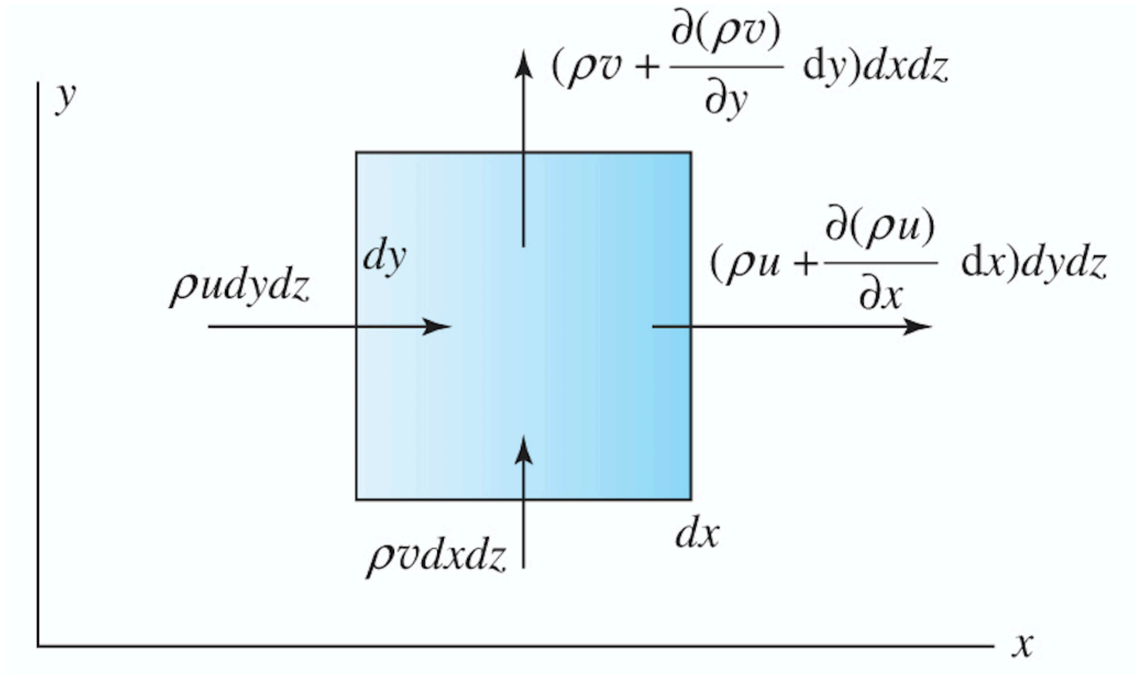
THE DIFFERENTIAL FORMS OF THE FUNDAMENTAL LAWS

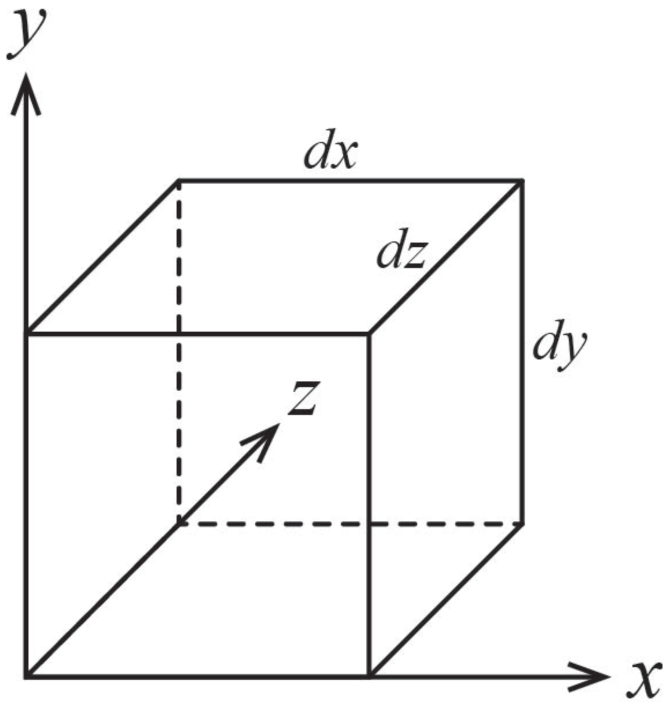
The basic equations considered in this chapter are the three laws of conservation for physical systems:

1. Conservation of mass (continuity)
2. Conservation of momentum (Newton's second law)
3. Conservation of energy (first law of thermodynamics)

CONSERVATION OF MASS
THE EQUATION OF CONTINUITY

Mass flow rate into the element in x- and y-direction is shown in the figure below.





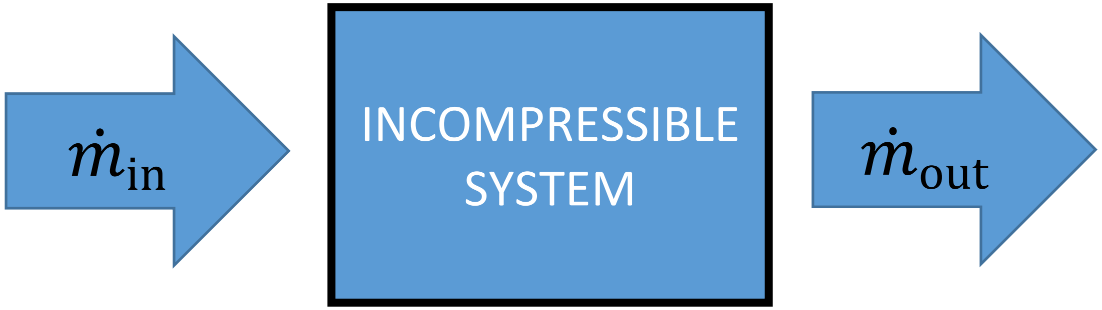
Mass flow rate $= \dot{m} = \rho A V$

$$\frac{\partial(\rho u)}{\partial x} = \rho \frac{\partial u}{\partial x} + u \cdot \frac{\partial \rho}{\partial x}$$

The net flux of mass entering the element equal to the rate of change of the mass of the element.

$$\dot{m}_{\text{in}} - \dot{m}_{\text{out}} = \frac{\partial}{\partial t} m_{\text{element}}$$

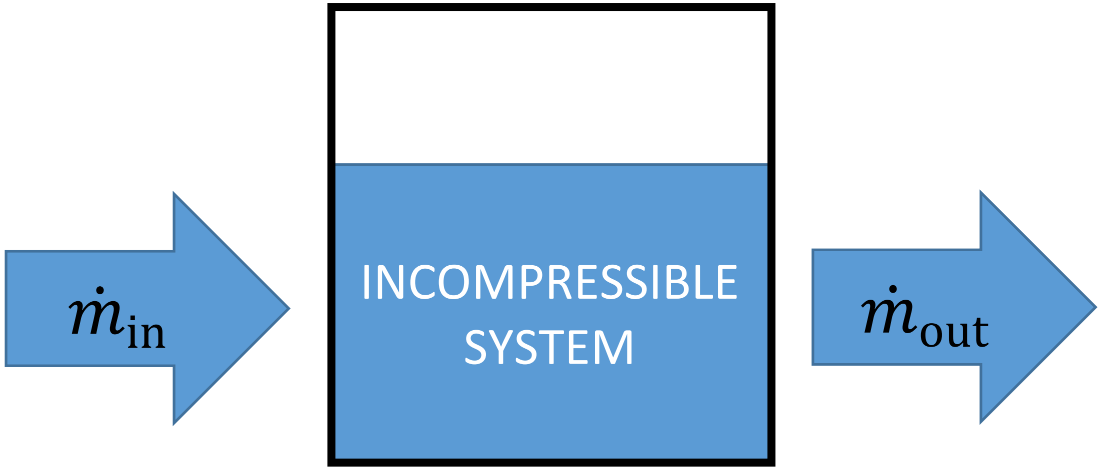
You may think like this:
Let say we have system of incompressible flow like this;



$$\dot{m}_{\text{in}} = \dot{m}_{\text{out}}$$

$$\dot{m}_{\text{in}} - \dot{m}_{\text{out}} = \frac{\partial}{\partial t} m_{\text{element}} = 0$$

Let say we have system of incompressible flow like this;

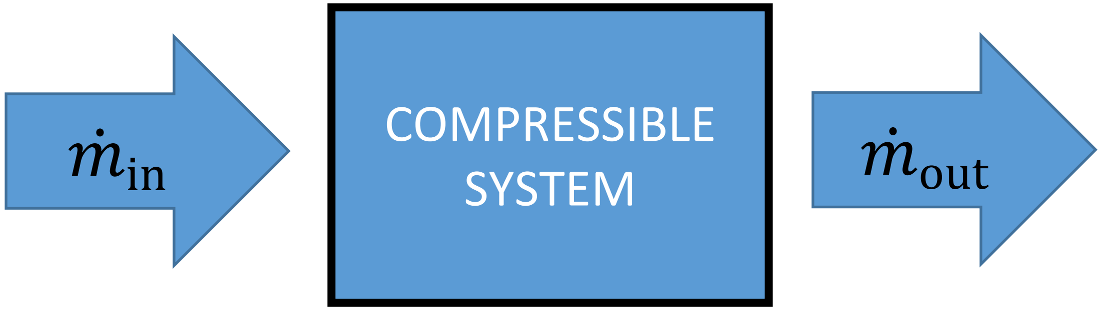


$$\dot{m}_{\text{in}} \neq \dot{m}_{\text{out}}$$

$$\dot{m}_{\text{in}} = \frac{\partial}{\partial t} m_{\text{element}} + \dot{m}_{\text{out}}$$

$$\dot{m}_{\text{in}} - \dot{m}_{\text{out}} = \frac{\partial}{\partial t} m_{\text{element}}$$

If we have system of compressible flow like this;



$$\dot{m}_{\mathrm{in}} \neq \dot{m}_{\mathrm{out}}$$

$$\dot{m}_{\mathrm{in}} = \frac{\partial}{\partial t} m_{\mathrm{element}} + \dot{m}_{\mathrm{out}}$$

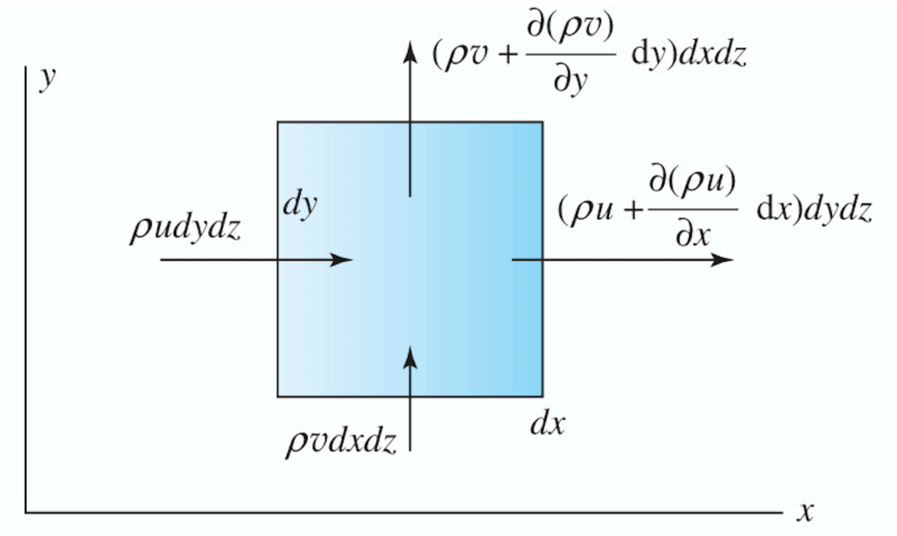
$$\dot{m}_{\mathrm{in}} - \dot{m}_{\mathrm{out}} = \frac{\partial}{\partial t} m_{\mathrm{element}}$$

The net flux of mass entering the element equal to the rate of change of the mass of the element.

$$\dot{m}_{\text{in}} - \dot{m}_{\text{out}} = \frac{\partial}{\partial t} m_{\text{element}}$$

$$\text{Mass flow rate} = \text{density} \times \text{velocity} \times \text{cross section area}$$

$$\rho u dydz + \rho v dxdz - \left(\rho u + \frac{\partial(\rho u)}{\partial x} dx\right) dydz - \left(\rho v + \frac{\partial(\rho v)}{\partial y} dy\right) dxdz = \frac{\partial}{\partial t}(\rho dxdydz)$$



$$\rho = \frac{m}{V}$$

$$m = \rho . V$$

Simplifying the above expression:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial \rho}{\partial t} = 0$$

$$\frac{\partial(\rho u)}{\partial x} = \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x}$$

If the z-direction is exist, it will become:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

Then, the differential continuity equation can be written as:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

This is the most general form of the differential continuity equation expressed using rectangular coordinates.

For the case of incompressible flow, a flow in which density of a fluid particle does not change as it travels along, the continuity equation becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Assume that we are discussing only 2-D coordinate, and there is no changes in density (incompressible), we might express the continuity equation as follows:

$$\frac{du}{dx} + \frac{dv}{dy} = 0$$

EQUATION OF CONTINUITY

Cartesian coordinates:

The continuity equation	$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$
We can simplify it	$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$
For 3D incompressible flow	$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$
For 2D incompressible flow	$\frac{du}{dx} + \frac{dv}{dy} = 0$

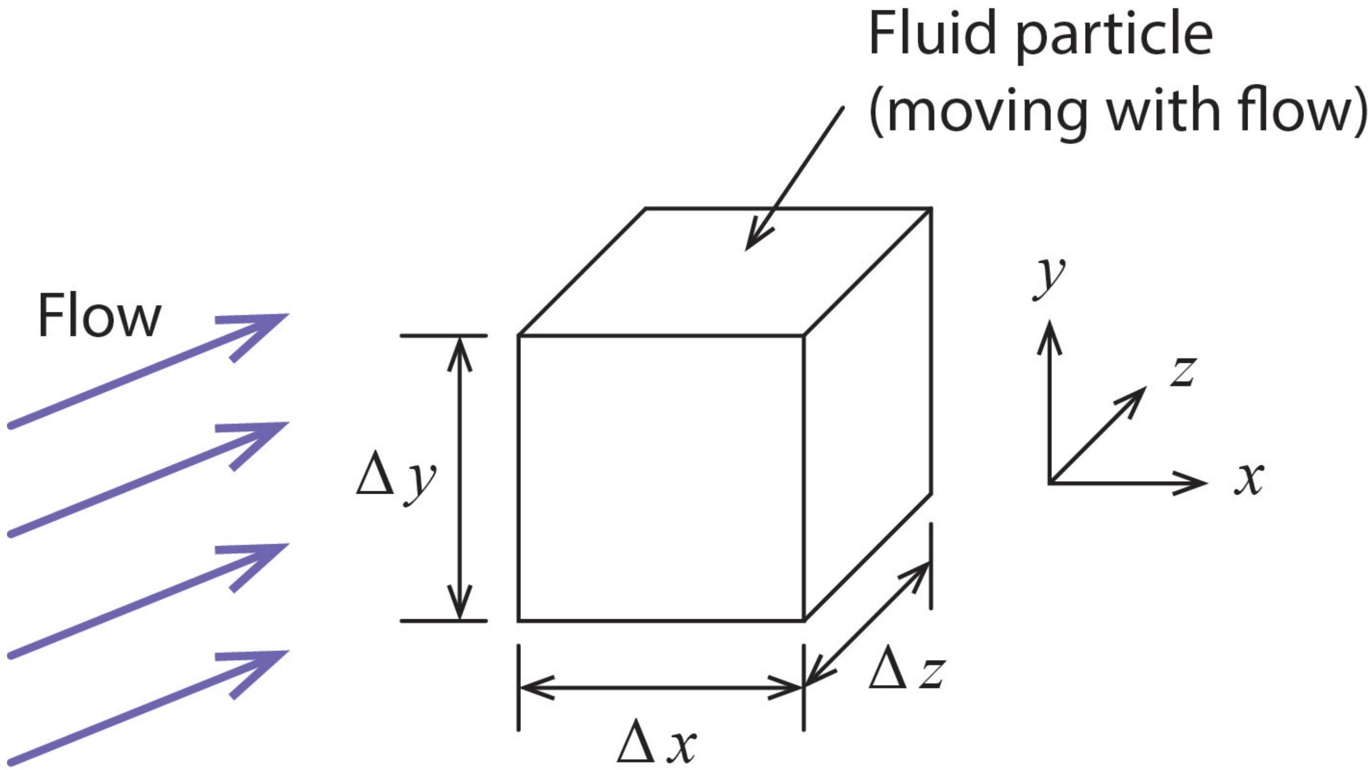
CONSERVATION OF MOMENTUM
THE NAVIER-STOKES EQUATIONS

The Navier-Stokes equation is widely used in both theory and in application. The Navier-Stokes equation represents Newton's second law of motion as applied to viscous flow of a Newtonian fluid. In this notes, we assume incompressible flow and constant viscosity.

Similar to continuity equations, there are multiple ways to derive the Navier-Stokes equation. This note shows how to derive the equation by starting with a fluid particle and applying Newton's second law. Thus, the result will be the non-conservation form of the equation. Because the derive is complex, we omit some of the technical details.

Step 1: Select a fluid particle

Select a fluid particle in a flowing fluid. Imagine that the particle has the shape of a cube. Assume the dimensions are infinitesimal and that the particle is at the position (x, y, z) at the instant shown.



Step 2: Apply Newton's second law

Regarding the forces, the two categories are body forces and surface forces. The only possible surface forces are the pressure force $\left(F_p\right)$and the shear force (F_s). Assume that the only body force is the weight (W).

$$F = ma$$

$$\text{Sum of forces on a particle} = (\text{ mass }) \times (\text{ acceleration })$$

$$\text{Body force} + \text{Surface force} = ma = \rho\forall a$$

$$(\forall = \text{volume})$$

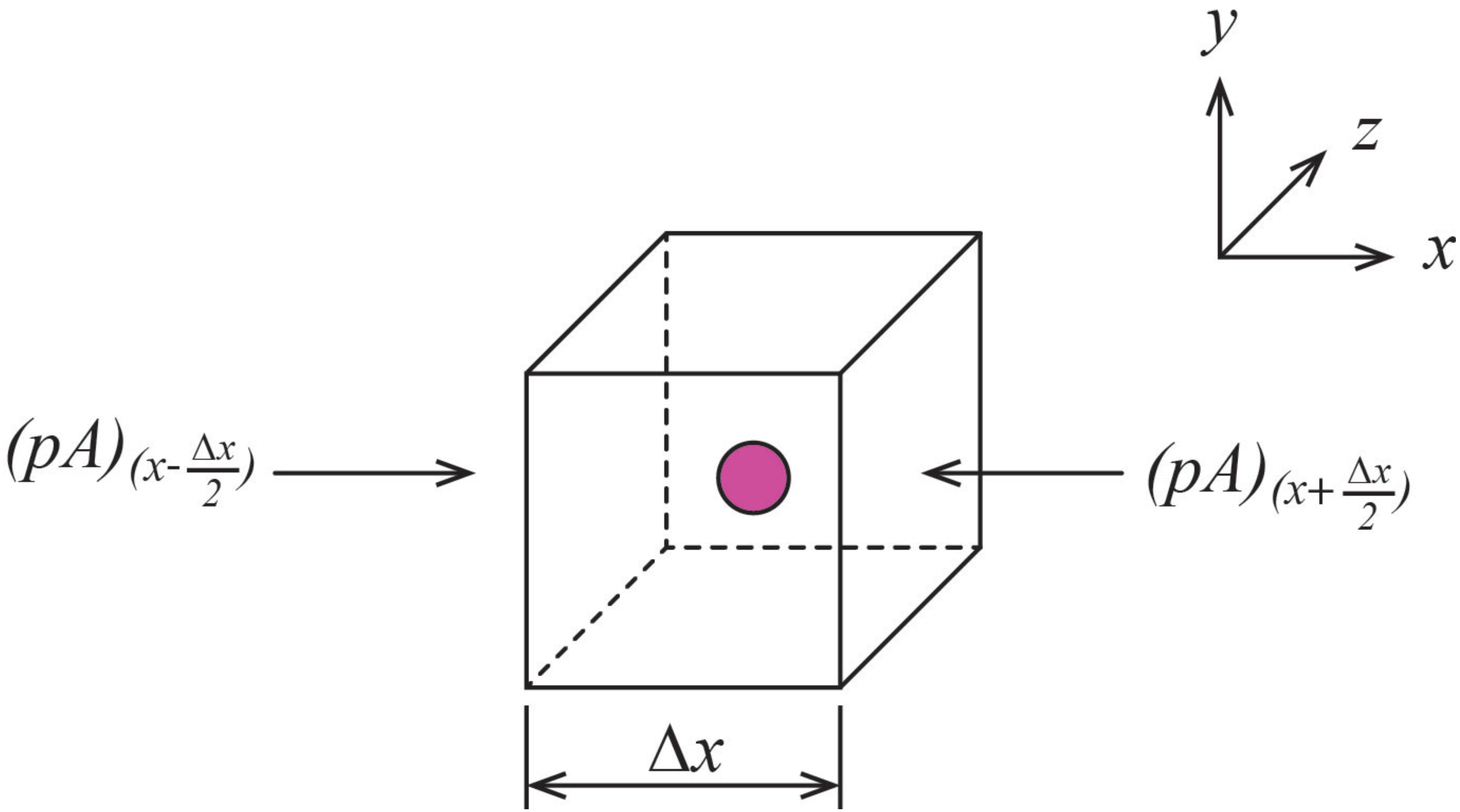
$$W + F_P + F_S = \rho\forall\frac{dv}{dt}$$

$$(W = mg = \rho\forall g)$$

$$\rho\forall g + F_P + F_S = \rho\forall\frac{dv}{dt}$$

Step 3: Analyze the pressure force

To begin, consider the forces on the x-faces of the particle.



The net force due to pressure on the x-faces is:

$$F_P = pA$$

$$F_{p-x} = \left(p_{x-\frac{\Delta x}{2}}(A) - p_{x+\frac{\Delta x}{2}}(A)\right) \cdot i$$

$$= \left(p_{x-\frac{\Delta x}{2}} - p_{x+\frac{\Delta x}{2}}\right) \cdot \Delta x \cdot \Delta y \cdot i$$

Simplify above equation by applying a Taylor series expansion (twice) and neglecting higher order term to give:

$$F_{P-x} = \frac{\partial p}{\partial x}(\Delta x \Delta y \Delta z) i$$

Repeat this process for the y-faces and z-faces, and combine results to give:

$$F_{P-all} = -\left(\frac{\partial p}{\partial x}(\Delta x \Delta y \Delta z) i + \frac{\partial p}{\partial y}(\Delta x \Delta y \Delta z) j + \frac{\partial p}{\partial z}(\Delta x \Delta y \Delta z) k\right)$$

Simplify it and then introduce vector notation to give:

$$F_P = -\left(\frac{\partial p}{\partial x} i + \frac{\partial p}{\partial y} j + \frac{\partial p}{\partial z} k\right)(\Delta x \Delta y \Delta z) = -\nabla p(\Delta x \Delta y \Delta z)$$

It reveals a physical interpretation of the gradient:

$$\begin{array}{r}\text{Gradient of the pressure} \\ \text{field at a point}\end{array} = \frac{\text{Net pressure force on a fluid particle}}{\text{Volume of the particle}}$$

Step 4: Analyze the shear force

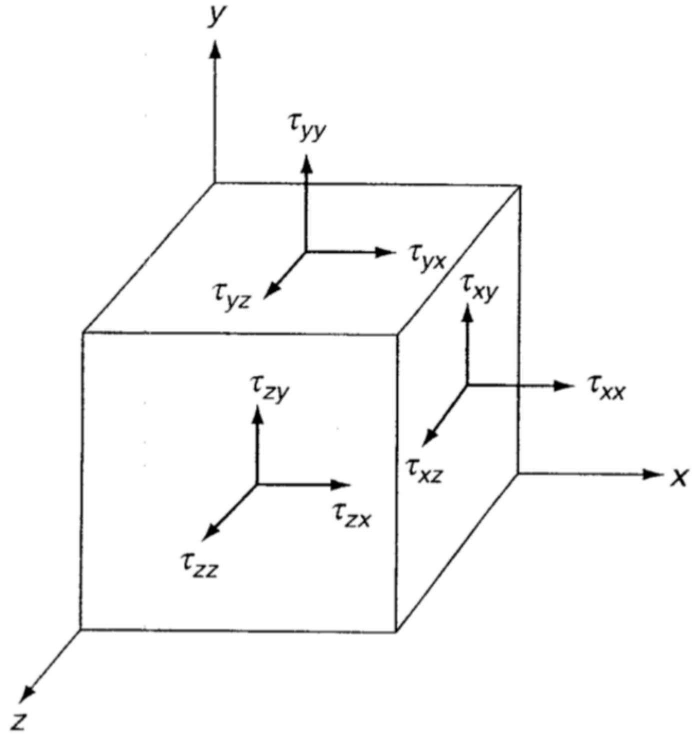
The shear force is the net force on the fluid particle due to shear stresses. Shear stress is caused by viscous effects and is represented mathematically as in Figure below. This figure shows that each face of the fluid particle has three (3) stress components. For example, the positive x-faces has three stress components, which are τ_{xx} , τ_{xy} and τ_{xz} . The double subscript notation describes the direction of the stress component and the face on which the component acts.

For example:

- τ_{xx} is the shear stress on the x-faces in the x-direction
- τ_{xy} is the shear stress on the x-faces in the y-direction
- τ_{xz} is the shear stress on the x-faces in the z-direction

Shear stress is a type of mathematical entity called a second order tensor. A tensor is analogous to but more general than a vector.

Example: A zeroth order tensor is a scalar, a first order tensor is a vector. A second order tensor has magnitude, direction and orientation (where orientation describes which face the stress acts on)



τ_{ij} = stress in the j direction on a face normal to the i axis

To find the net shear force on the particle, each stress component is be multiplied by area, and the forces are added. Then, a Taylor series expansion is applied and the result is that:

$$F_{Shear} = F_S = \begin{bmatrix} F_{x,shear} \\ F_{y,shear} \\ F_{z,shear} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{\partial \tau_{xx}}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \tau_{xz}}{\partial x} \right) \\ \left(\dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{yy}}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial y} \right) \\ \left(\dfrac{\partial \tau_{zx}}{\partial z} + \dfrac{\partial \tau_{zy}}{\partial z} + \dfrac{\partial \tau_{zz}}{\partial z} \right) \end{bmatrix} (\Delta x \Delta y \Delta z)$$

This can be written in invariant notation as:

$$F_{Shear} = F_S = (\nabla \cdot \tau)\forall = \big(\mathrm{div}(\tau)\big)\forall$$

where the terms on the right side represent the divergence of the stress tensor times the volume of the fluid particle.

It reveals the physics of the divergence when it operates on the stress tensor. Note that this is the third physical interpretation of the divergence operator. This is because the physics of a mathematical operator depend on the context in which the operator is used.

$$\begin{array}{r}\text{Divergence of the}\\ \text{stress tensor}\end{array} = \frac{\text{Net shear force on a fluid particle}}{\text{Volume of the particle}}$$

Step 5: Combine terms

Substitute the shear force and pressure force into Newton's second law of motion. Then, divide by the volume of the fluid particle to give:

$$
\begin{aligned}
ma &= F \\
\rho(\forall)\frac{dv}{dt} &= \rho\forall g + F_P + F_S \\
&= \rho\forall g - \nabla p(\Delta x\Delta y\Delta z) + (\nabla\cdot\tau)\forall \\
&= \rho g(\forall) - \nabla p(\forall) + (\nabla\cdot\tau)(\forall) \\
&\text{Divide with volume, } (\forall) \\
\rho\frac{dv}{dt} &= \rho g - \nabla p + \nabla\tau
\end{aligned}
$$

This is the differential form of the linear momentum equation without any assumption about the nature of the fluid. The next step involves modifying this equation to that it applies to a Newtonian fluid.

Step 6: Assume a Newtonian fluid

Stokes in 1845 figured out a way to write the stress tensor in terms of the rate-of-strain tensor of the flowing fluid. The details are omitted here. After Stokes' results are introduced, assume constant density and viscosity, above equation becomes:

$$\rho \frac{dv}{dt} = \rho g - \nabla p + \nabla \tau$$

Above equation can be specifically written as:

$x - momentum$	$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$
$y - momentum$	$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$
$z - momentum$	$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$

It may be noted that the last three convective terms on the RHS of above mention equations make it highly non-linear and complicates the general analysis. A simplification is possible for considering an incompressible flow of Newtonian fluid where the viscous stresses are proportional to the element strain rate and coefficient of viscosity (μ). For an incompressible flow, the shear terms may be written:

$\tau_{xx} = 2\mu \frac{\partial u}{\partial x}$	$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial x}{\partial x}\right)$
$\tau_{yy} = 2\mu \frac{\partial v}{\partial y}$	$\tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$
$\tau_{zz} = 2\mu \frac{\partial w}{\partial z}$	$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$

Thus, the differential momentum equation for Newtonian fluid with constant density and viscosity is given by:

$x - momentum$	$\rho \frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$
$y - momentum$	$\rho \frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$
$z - momentum$	$\rho \frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$

It is a second order, non-linear partial differential equation and is known as Navier-Stokes equation. In vector form, it may be represented as:

$$\rho \frac{DV}{Dt} = \rho g - \nabla p + \mu \nabla^2 V$$

where $\nabla^2 V$ is a mathematical operator that is called the Laplacian of the velocity field.

As a conclusion, this is the Navier-Stokes equation.

$$\boxed{\rho \frac{DV}{Dt} = \rho g - \nabla p + \mu \nabla^2 V}$$

Step 7 : Interpret the physics

The physics of the Navier-Stokes equation are:

$\rho \dfrac{DV}{Dt}$	=	ρg	−	∇p	+	$\mu \nabla^2 V$
Mass of the particle times acceleration of the particle divided by the volume of the particle.		Weight of the particle divided by its volume.		Net pressure force on the particle divided by its volume.		Net shear force on the particle divided by its volume.

Note the dimensions and units:

$$\text{Dimension} = \frac{\text{Force}}{\text{Volume}} = \frac{N}{m^3} = \frac{kg}{m^2 \cdot s^2}$$

Navier-Stokes equation (constant properties) for Cartesian coordinates is:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

EQUATION OF CONTINUITY

Cartesian coordinates:

$$0 = \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z}$$

Polar coordinates:

$$0 = \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho u_r r) + \frac{\partial}{\partial z}(\rho u_z) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta)$$

Navier-Stokes equation (constant properties) for Cartesian coordinates is:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{Du}{Dt} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon(\nabla^2)u$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) = \rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} = g_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{Du}{Dt} = g_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon(\nabla^2)v$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) = \rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} = g_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{Dw}{Dt} = g_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\upsilon(\nabla^2)w$$

CONSERVATION OF ENERGY

Recall the integral relation of energy equation for a fixed control volume:

$$\dot{Q} - \dot{W}_S - \dot{W}_V = \frac{\partial E}{\partial t} = \frac{\partial}{\partial t}\left(\int_{CV} e\rho d\forall\right) + \int_{CS} \left(e + \frac{p}{\rho}\right)\rho(V \cdot n)\, dA \qquad (1)$$

$\dot{Q}$ = Rate of heat energy added to the control volume

$\dot{W}_S$ = Time derivative of shaft work in the control volume

$\dot{W}_V$ = Time derivative of work done by viscous stress in the control volume

$\frac{\partial E}{\partial t}$ = Rate of change of energy in the system

CV = Control volume

CS = Control surface

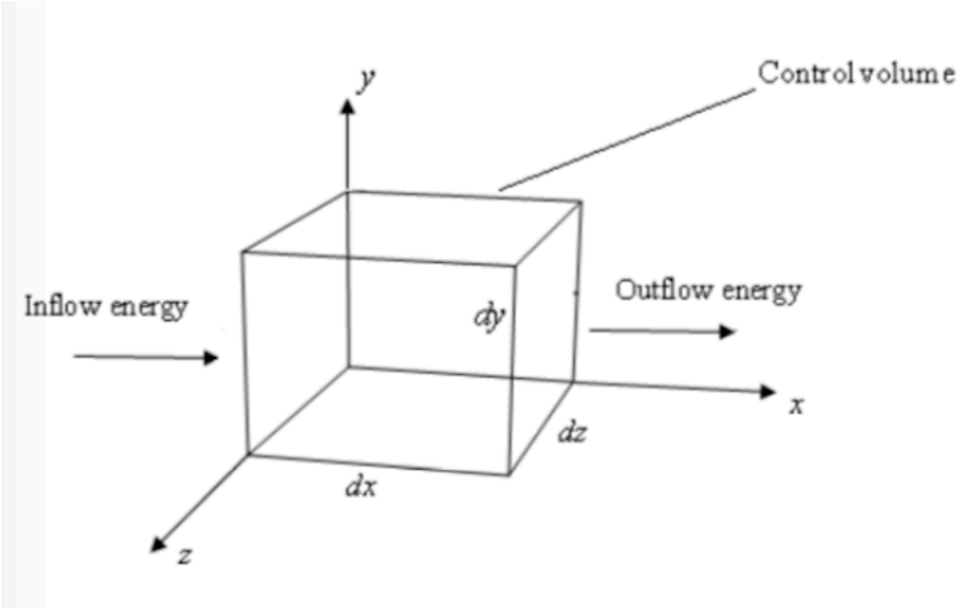


Figure 1

If the control volume happens to be an elemental system as shown in Figure 1, then there will be no shaft work term $\left(\dot{W}_S = 0\right)$.
Denoting the energy per unit volume as:

$$e = \hat{u} + \frac{1}{2}V^2 + gz$$

The net energy flow across the six control surface can be calculated as below:

Face	Inlet energy flow	Outlet energy flow
x	$\rho u\left(e+\frac{p}{\rho}\right)dydz$	$\left[\rho u\left(e+\frac{p}{\rho}\right)+\frac{\partial}{\partial x}(\rho u)\left(e+\frac{p}{\rho}\right)dx\right]dydz$
y	$\rho v\left(e+\frac{p}{\rho}\right)dxdz$	$\left[\rho v\left(e+\frac{p}{\rho}\right)+\frac{\partial}{\partial y}(\rho v)\left(e+\frac{p}{\rho}\right)dy\right]dxdz$
z	$\rho w\left(e+\frac{p}{\rho}\right)dxdy$	$\left[\rho w\left(e+\frac{p}{\rho}\right)+\frac{\partial}{\partial w}(\rho w)\left(e+\frac{p}{\rho}\right)dz\right]dxdy$

Hence, Eq.(1) can be written as:

$$\dot{Q}-\dot{W}_V=\left[\frac{\partial}{\partial t}\rho\left(e+\frac{p}{\rho}\right)+\frac{\partial}{\partial x}(\rho u)\left(e+\frac{p}{\rho}\right)+\frac{\partial}{\partial y}(\rho v)\left(e+\frac{p}{\rho}\right)+\frac{\partial}{\partial z}(\rho w)\left(e+\frac{p}{\rho}\right)\right]dxdydz \qquad (2)$$

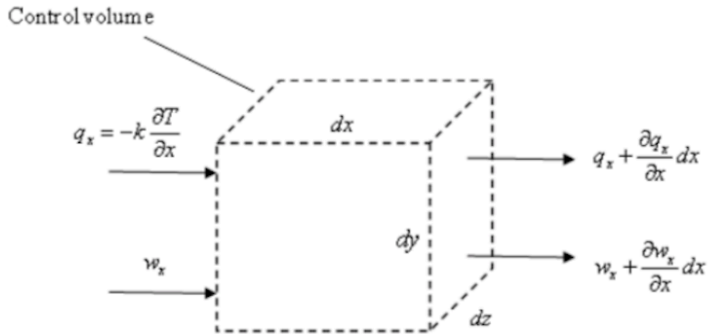
With the help of continuity equation and similar analogy considered during the derivation of momentum equation, Eq.(2) can be simplified as:

$$\dot{Q} - \dot{W}_V = \left[\rho \frac{de}{dt} + V \cdot (\nabla p)\right] dxdydz \tag{3}$$

If one considers the energy transfer as heat $(\dot{Q})$ through pure conduction, the Fourier's law of heat conduction can be applied to the elemental system.

$$q = -k\nabla T \tag{4}$$

Where k is the thermal conductivity of the fluid.



The heat flow passing through x-face is shown in Figure 2, and for all the six faces, it is summarized in the following table:

Face	Inlet energy flow	Outlet energy flow
x	$q_x dydz$	$\left[q_x + \frac{\partial}{\partial x}(q_x)dx\right]dydz$
y	$q_y dxdz$	$\left[q_y + \frac{\partial}{\partial y}(q_y)dx\right]dxdz$
z	$q_z dxdy$	$\left[q_z + \frac{\partial}{\partial z}(q_z)dx\right]dxdy$

The net heat flux can be obtained by the difference in inlet and outlet heat fluxes.

$$\begin{aligned}\dot{Q} &= -\left[\frac{\partial}{\partial x}(q_x) + \frac{\partial}{\partial y}\left(q_y\right) + \frac{\partial}{\partial z}(q_z)\right] dxdydz \\ &= -(\nabla \cdot q)dxdydz \\ &= \nabla \cdot (k\nabla T)\, dxdydz \end{aligned} \tag{5}$$

The rate of work done by the viscous stresses on the left x-faces as shown in the Figure 2 is given by:

$$\begin{aligned}\dot{W}_V &= -w_x dydz \\ &= -\left(u\tau_{xx} + v\tau_{xy} + w\tau_{xz}\right)dydz \end{aligned} \tag{6}$$

In the similar manner, the net viscous rates are obtained and is given by:

$$\dot{W}_V = -\left[\frac{\partial}{\partial x}\left(u\tau_{xx} + v\tau_{yy} + w\tau_{zz}\right) + \frac{\partial}{\partial y}\left(u\tau_{yx} + v\tau_{yy} + w\tau_{yz}\right) + \frac{\partial}{\partial z}\left(u\tau_{zx} + v\tau_{zy} + w\tau_{zz}\right)\right] dxdydz$$

$$= -\nabla\left(V \cdot \tau_{ij}\right) dxdydz \tag{7}$$

Then, substitute Eq.(5) and Eq.(7) into Eq.(3),

$$\rho\frac{de}{dt} + V \cdot \nabla p = \nabla \cdot (k\nabla T) + \nabla \cdot \left(V \cdot \tau_{ij}\right) \tag{8}$$

The second term in the RHS of Eq.(8) can be written in the following term,

$$\nabla \cdot \left(V \cdot \tau_{ij}\right) = V \cdot \left(\nabla \cdot \tau_{ij}\right) + \Phi \tag{9}$$

Here, Φ is known as the viscous-dissipation function. For Newtonian incompressible viscous fluid, this function as the following form.

$$\Phi = \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right] \qquad (10)$$

Since all the terms in Eq.(10) are quadratic, so the viscous dissipation terms are always positive, the the flow always tends to lose its available energy due to dissipation.

When Eq.(9) is used in Eq.(8), simplified using linear-momentum equation and the terms are rearranged, then the general form of energy equation is obtained for Newtonian viscous fluid.

$$\rho\frac{du}{dt} + p(\nabla \cdot V) = \nabla \cdot (k\nabla T) + \Phi \qquad (11)$$

For analysis point of view, the following valid approximations can be made for Eq.(11)

$$du \approx c_V dT$$

c_v , μ , k and ρ are constant

$$\rho c_v \frac{dT}{dt} = k\nabla^2 T + \Phi \tag{13}$$

Where,

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

The specific heats of gases are given as c_p and c_v at constant pressure and constant volume respectively while solids and liquids are having only single value for specific heat.