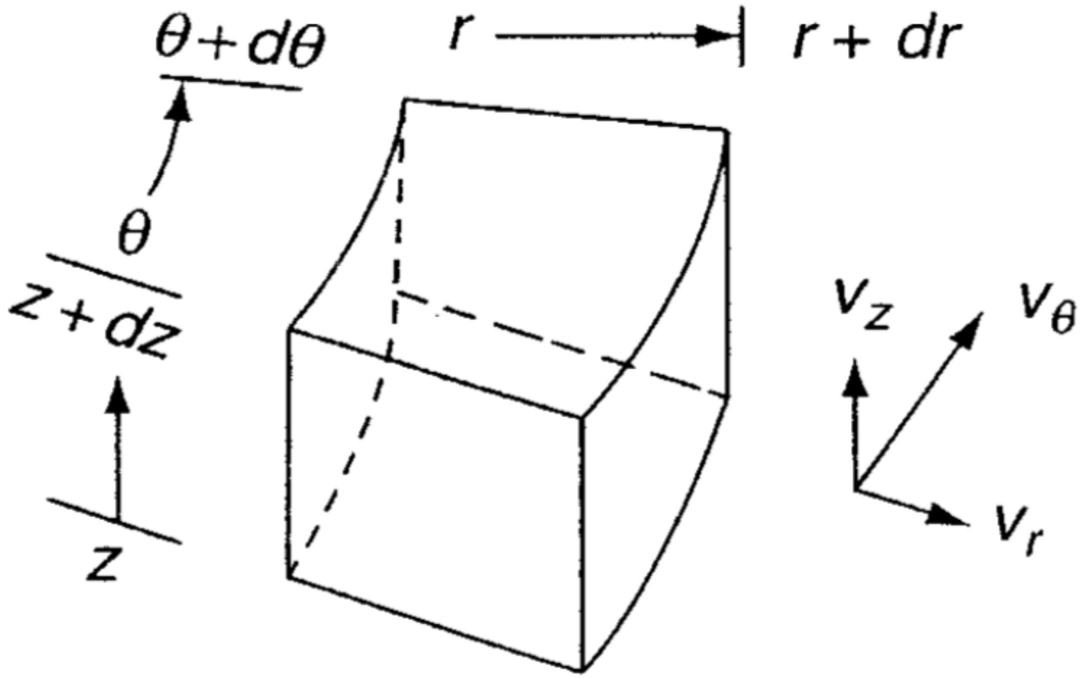
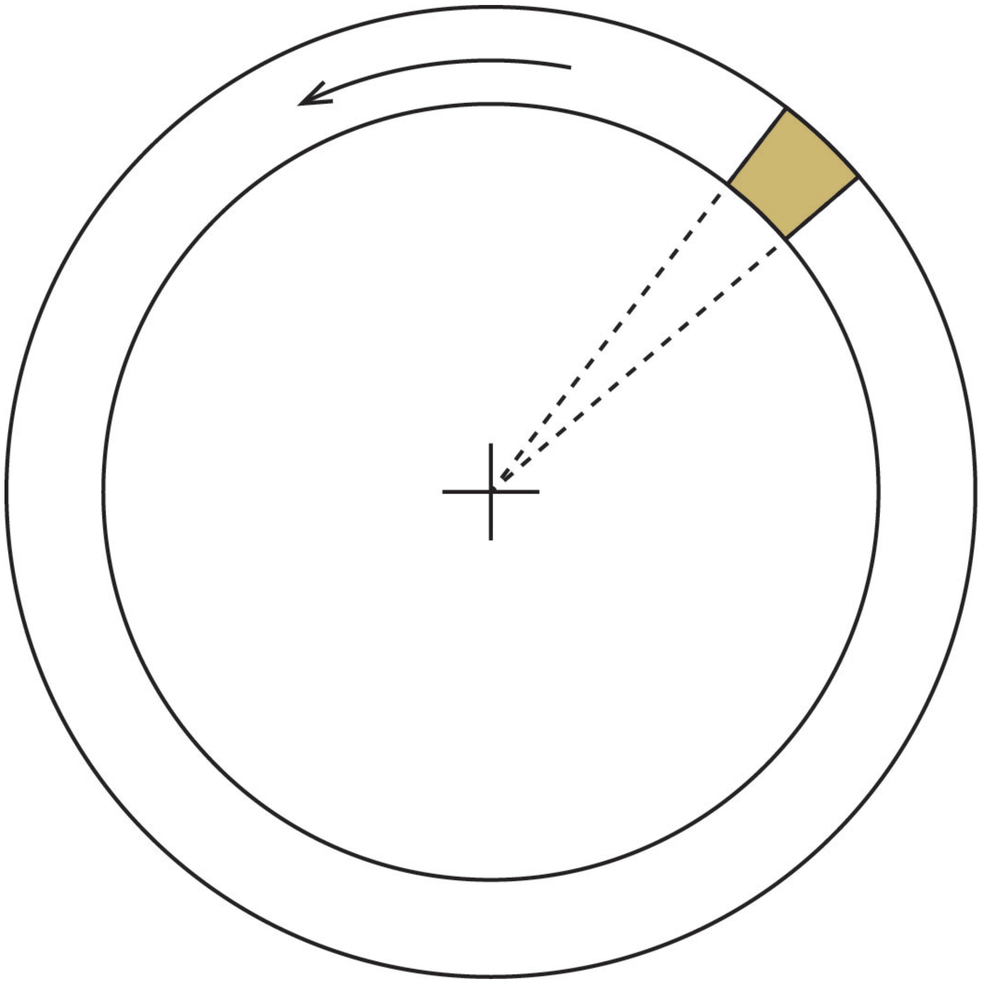
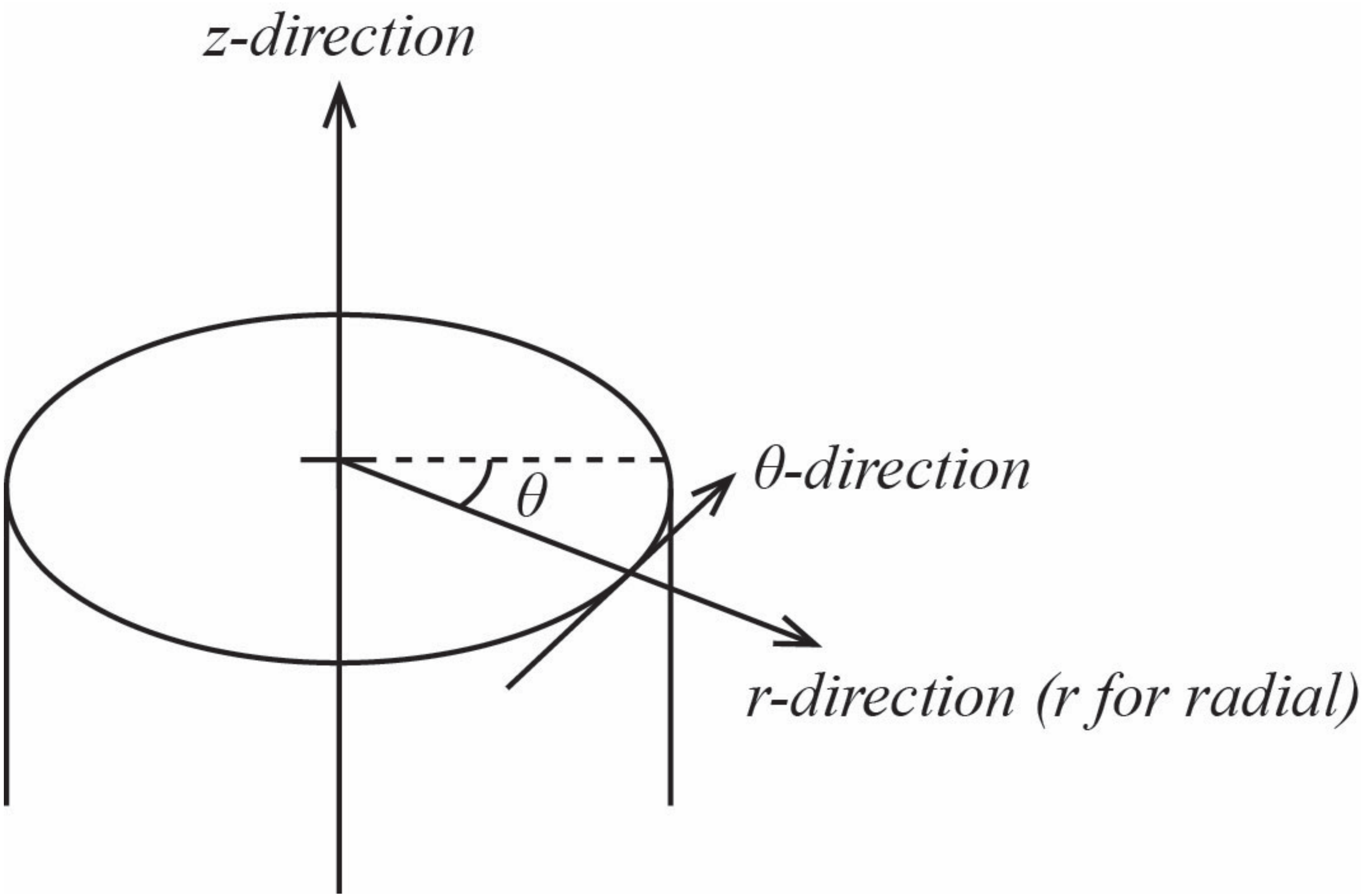
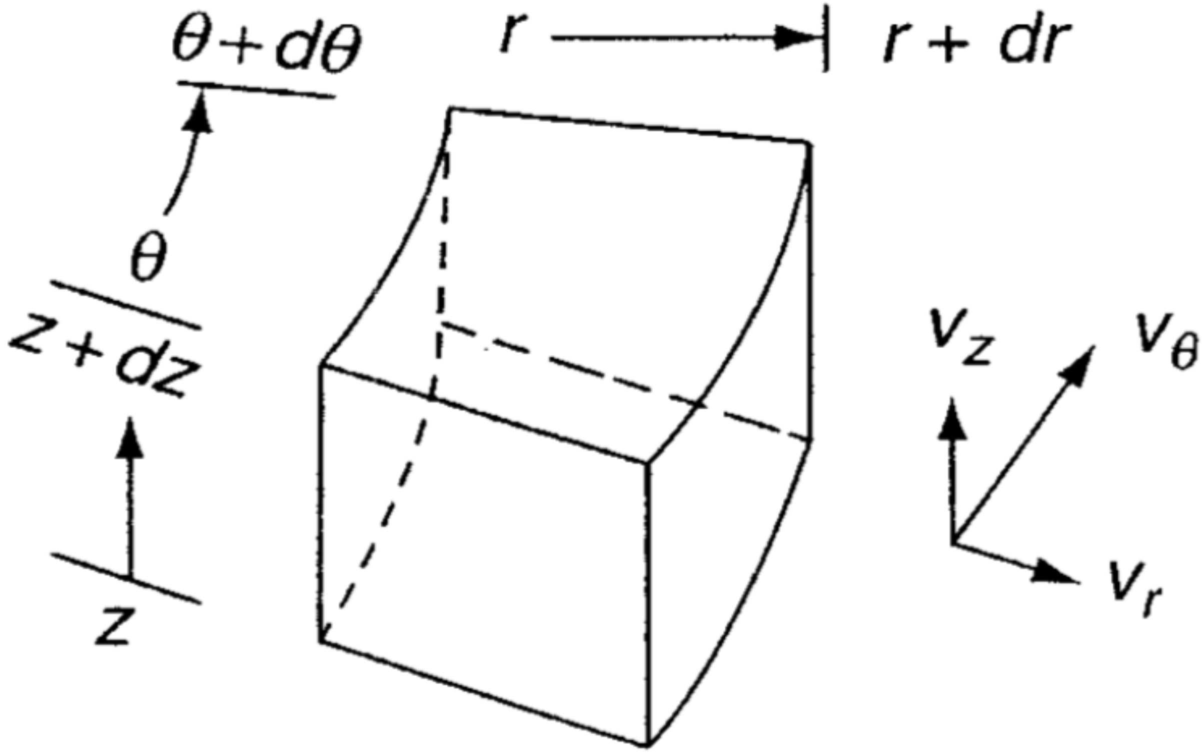
CONTINUITY EQUATION IN CYLINDRICAL (POLAR) COORDINATES.

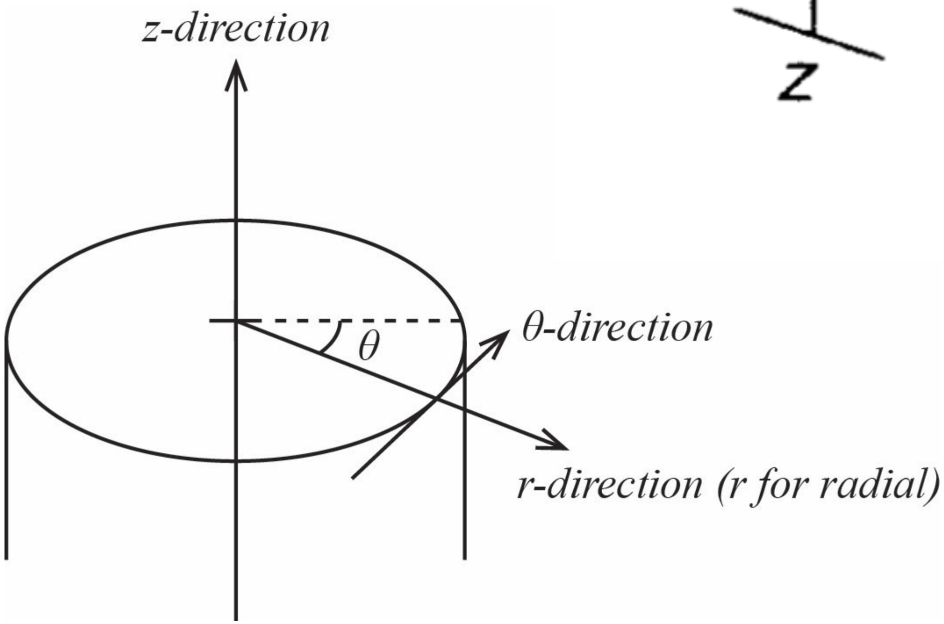
By consideration of the cylindrical elemental control volume as shown below, use the conservation of mass to derive the continuity equation in cylindrical coordinates.






z-direction
θ
θ-direction
r-direction (r for radial)


θ + dθ
r
r + dr
θ
z + dz
z
v_z
v_θ
v_r


z-direction
θ
θ-direction
r-direction (r for radial)

CONTINUITY EQUATION

The net flux of mass entering the element equal to the rate of change of the mass of the element.

$$\dot{m}_{in} - \dot{m}_{out} = \frac{\partial}{\partial t} m_{element}$$

$$\dot{m} = \text{mass flow rate} = \rho A V$$

$$V = \text{velocity of fluid}$$

(a) Mass flow rate, direction of v_r :

$$\dot{m}_{in} - \dot{m}_{out} = \rho(rd\theta dz)(v_r) - \left(\rho v_r + \frac{\partial}{\partial r}(\rho v_r)dr\right)(r+dr)d\theta dz$$

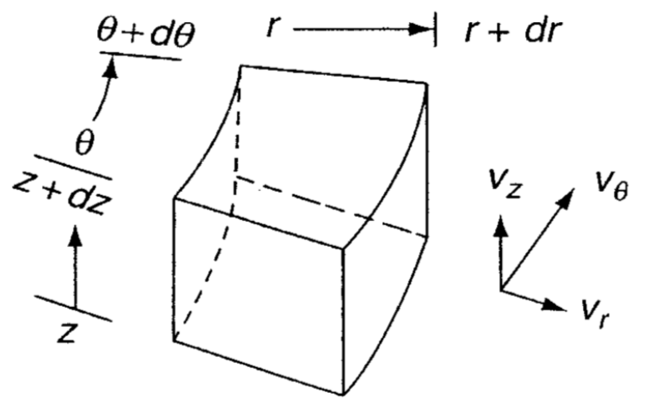
$$= \rho(rd\theta dz)(v_r) - \left[\rho v_r r + \rho v_r dr + \frac{\partial}{\partial r}(\rho v_r)rdr + \frac{\partial}{\partial r}(\rho v_r)drdr\right]d\theta dz$$

$$(drdr = 0, \text{ too small})$$

$$= \rho v_r r d\theta dz - \left[\rho v_r r + \rho v_r dr + \frac{\partial}{\partial r}(\rho)rdr\right]d\theta dz$$

$$= \rho v_r r d\theta dz - \left[\rho v_r r + \frac{\partial}{\partial r}(\rho v_r r)dr\right]d\theta dz$$

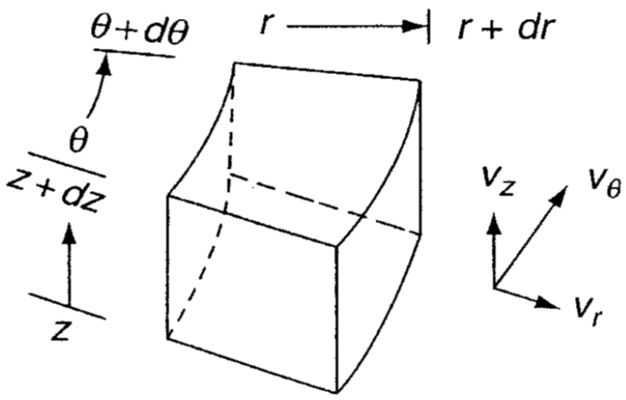
$$= -\frac{\partial}{\partial r}(\rho v_r r)drd\theta dz$$



(b) Mass flow rate, direction of v_z:

$$\dot{m}_{in} - \dot{m}_{out} = \rho v_z r d\theta dr - \left(\rho v_z + \frac{\partial}{\partial z}(\rho v_z)dz\right) r d\theta dr$$

$$= -\frac{\partial}{\partial z}(\rho v_z) r d\theta dr dz$$

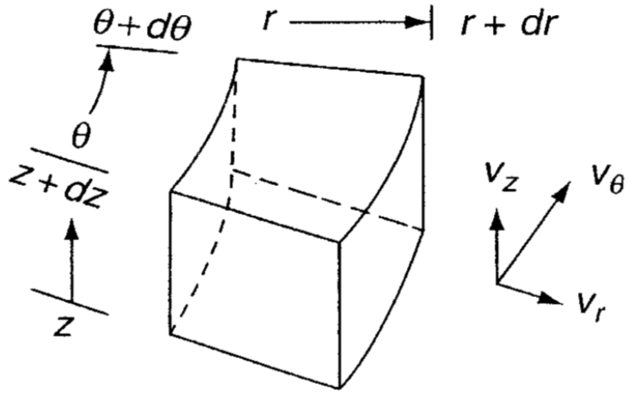


(c) Mass flow rate, direction of v_θ :

$$\dot{m}_{in} - \dot{m}_{out} = \rho v_\theta dz dr - \left(\rho v_\theta + \frac{\partial}{\partial \theta}(\rho v_\theta)d\theta\right) dr dz$$

$$= -\frac{\partial}{\partial \theta}(\rho v_\theta) d\theta dr dz$$

(d)



$$mass = \rho \forall \qquad (\forall = \text{Volume})$$

$$= \rho(rd\theta drdz)$$

$$\frac{\partial}{\partial t} m = \frac{\partial}{\partial t}(\rho rd\theta drdz)$$

$$\dot{m}_{in} - \dot{m}_{out} = \frac{\partial}{\partial t} m_{element}$$

$$-\frac{\partial}{\partial r}(\rho v_r r)drd\theta dz - \frac{\partial}{\partial z}(\rho v_z)rd\theta drdz - \frac{\partial}{\partial \theta}(\rho v_\theta)d\theta drdz = \frac{\partial}{\partial t}(\rho rd\theta drdz)$$

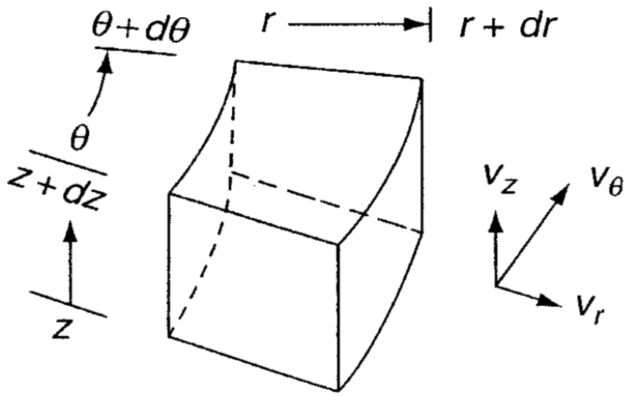
(divide with $d\theta drdz$)

$$-\frac{\partial}{\partial r}(\rho v_r r) - \frac{\partial}{\partial z}(\rho v_z)r - \frac{\partial}{\partial \theta}(\rho v_\theta) = \frac{\partial}{\partial t}(\rho r)$$

$$= \rho \frac{\partial r}{\partial t} + r\frac{\partial \rho}{\partial t}$$

$$\left(\frac{\partial r}{\partial t} = 0,\ \text{No changes of } \ r \ \text{ regarding to the time}\right)$$

$$0 \ = r\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho v_r r) + \frac{\partial}{\partial z}(\rho v_z) r + \frac{\partial}{\partial \theta}(\rho v_\theta)$$



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$$0 \ = \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho v_r r) + \frac{1}{r}\frac{\partial}{\partial z}(\rho v_z) r + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)$$

$$0 \ = \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho v_r r) + \frac{\partial}{\partial z}(\rho v_z) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)$$

This is the compressible equation of continuity in cylindrical polar coordinates.

For 2-D incompressible fluid, there is no changes in density and velocity for z-direction is zero.

Thus, continuity equation becomes;

$$0 = \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho v_r r) + \frac{\partial}{\partial z}(\rho v_z) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(v_r r) + \frac{\partial}{\partial z}(v_z) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)$$

$$0 = \frac{v_r}{r} + \frac{\partial (v_r)}{\partial r} + \frac{1}{r}\frac{\partial (v_\theta)}{\partial \theta}$$

Sometimes, it can be written as;

$$0 = \frac{u'}{r} + \frac{\partial u'}{\partial r} + \frac{1}{r}\frac{\partial v'}{\partial \theta}$$

EQUATION OF CONTINUITY

Cartesian coordinates:

$$0 = \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}$$

Polar coordinates:

$$0 = \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho u_r r) + \frac{\partial}{\partial z}(\rho u_z) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta)$$

Navier-Stokes equation (constant properties) for cylindrical (polar) coordinates is:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$r - axis$:

$$\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = g_r - \frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} + \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = g_r - \frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} + \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$\theta - axis$:

$$\frac{Du_\theta}{Dt} - \frac{u_r u_\theta}{r} = g_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} - \frac{u_r u_\theta}{r} = g_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right)$$

$z - axis$:

$$\frac{Du_z}{Dt} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu(\nabla^2 u_z)$$

$$\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right)$$

Note:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{1}{r}\left[r\frac{\partial}{\partial r}\left(\frac{\partial u_z}{\partial r}\right) + \frac{\partial u_z}{\partial r}\left(\frac{\partial r}{\partial r}\right)\right] = \frac{1}{r}\left[r\frac{\partial^2 u_z}{\partial r^2} + \frac{\partial u_z}{\partial r}\right]$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}$$