Chapter 4

The Integral Forms of the Fundamental Laws

4.2 The Three Basic Laws

- The integral quantities in fluid mechanics are contained in the three laws:
  - Conservation of Mass
  - First Law of Thermodynamics
  - Newton's Second Law

- They are expressed using a Lagrangian description in terms of a system (fixed collection of material particles).

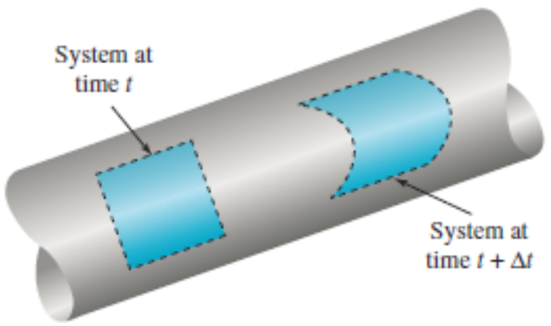


Figure 4.1 Example of a system in fluid mechanics.

4.2 The Three Basic Laws

- **CONSERVATION OF MASS: Mass of a system remains constant.**

$$\frac{D}{Dt}\int_{sys} \rho \, d\mathcal{V} = 0$$

Integral form of the mass-conservation equation. ρ = Density; d$\mathcal{V}$ = Volume occupied by the particle

- **FIRST LAW OF THERMODYNAMICS: Rate of heat transfer to a system minus the rate at which the system does work equals the rate at which the energy of the system is changing.**

$$\dot{Q} - \dot{W} = \frac{D}{Dt}\int_{sys} e\rho \, d\mathcal{V}$$

Specific energy (e): Accounts for kinetic energy per unit mass ($0.5V^2$), potential energy per unit mass (gz), and internal energy per unit mass ($\tilde{u}$).

4.2 The Three Basic Laws

- **NEWTON'S SECOND LAW: Resultant force acting on a system equals the rate at which the momentum of the system is changing.**

$$\Sigma \mathbf{F} = \frac{D}{Dt}\int_{\text{sys}} \mathbf{V}\rho \, d\forall$$

In an inertial frame of reference.

- **Moment-of-Momentum Equation: Resultant moment acting on a system equals the rate of change of the angular momentum of the system.**

$$\Sigma \mathbf{M} = \frac{D}{Dt}\int_{\text{sys}} \mathbf{r} \times \mathbf{V}\rho \, d\forall$$

4.2 The Three Basic Laws

- **Control Volume**: A region of space into which fluid enters and/or from which fluid leaves.

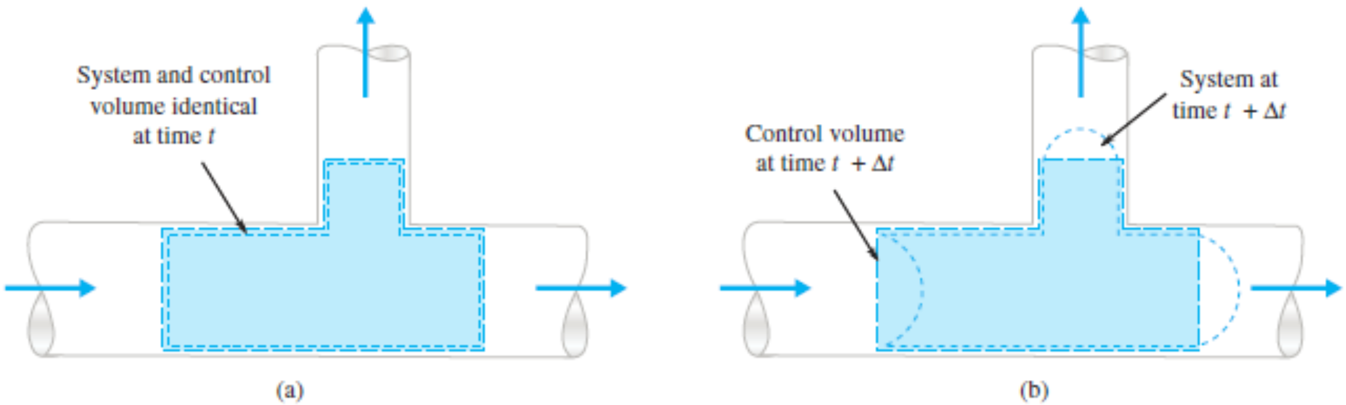


Figure 4.2 Example of a fixed control volume and a system: (a) time t; (b) time $t + \Delta t$.

4.3 System-to-Control-Volume Transformation

- Interested in the time rate of change of an extensive property to be expressed in terms of quantities related to a control volume.
  - Involves fluxes of an extensive property in and out of a control volume.
  - **Flux** is the measure of the rate at which an extensive property crosses an area.

4.3 System-to-Control-Volume Transformation

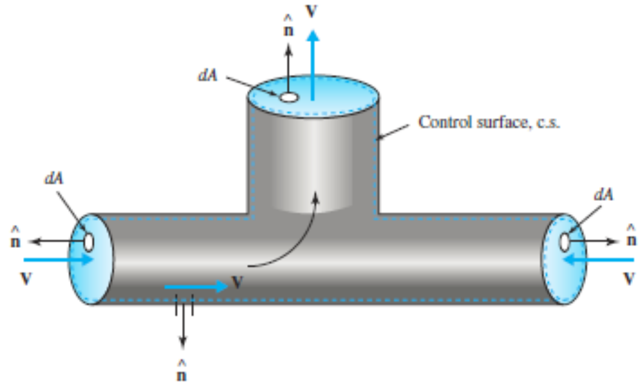


Figure 4.3 Illustration showing the flux of an extensive property.

Control surface: The surface area that completely encloses the control volume.

- The flux across an element dA is:

$$\text{flux across } dA = \eta\rho\hat{\mathbf{n}}\cdot\mathbf{V}\, dA$$

$\hat{n}$: Unit vector normal to dA (<u>always points out</u> of the control volume)
η: Intensive property

- Only the normal component of $\hat{n}\cdot$V contributes to this flux.
  - Positive component means a flux out of the volume.
  - Negative component indicates a flux into the volume.
  - If the net flux is positive: Flux out > flux in

4.3 System-to-Control-Volume Transformation

Reynolds Transport Theorem

- The Reynolds transport theorem is a system-to-control-volume transformation.

$$\frac{DN_{sys}}{Dt} = \frac{d}{dt}\int_{c.v.} \eta\rho \, dV\!\!\!\!/ + \int_{c.s.} \eta\rho\hat{\mathbf{n}}\cdot\mathbf{V}\, dA$$

- This is a Lagrangian-to-Eulerian transformation of the rate of change of an extensive quantity.
  - First part of integral: Rate of change of an extensive property in the control volume.
  - Second part of integral: Flux of the extensive property across the control surface (nonzero where fluid crosses the control surface).

4.3 System-to-Control-Volume Transformation

Reynolds Transport Theorem

$$\frac{DN_{sys}}{Dt} = \frac{d}{dt}\int_{c.v.} \eta\rho \; d\maltese + \int_{c.s.} \eta\rho\hat{\mathbf{n}}\cdot\mathbf{V}\, dA$$

- An equivalent form of the control volume is:

$$\frac{DN_{sys}}{Dt} = \int_{c.v.} \frac{\partial}{\partial t}(\rho\eta)\, d\maltese + \int_{c.s.} \eta\rho\hat{\mathbf{n}}\cdot\mathbf{V}\, dA$$

- The time derivative of the control volume is moved inside the integral:
  - For a fixed control volume, the limits on the volume integral are independent of time.

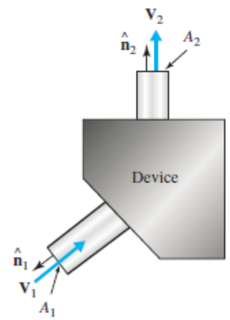
4.3 System-to-Control-Volume Transformation

4.3.1 Simplifications of the Reynolds Transport Theorem

$$\frac{DN_{sys}}{Dt} = \frac{d}{dt}\int_{c.v.} \eta\rho \, d\mathcal{V} + \int_{c.s.} \eta\rho\hat{\mathbf{n}}\cdot\mathbf{V}\,dA$$

$$\frac{DN_{sys}}{Dt} = \int_{c.v.} \frac{\partial}{\partial t}(\rho\eta)\,d\mathcal{V} + \int_{c.s.} \eta\rho\hat{\mathbf{n}}\cdot\mathbf{V}\,dA$$

- **Steady-state (time derivative is zero)**: $\frac{DN_{sys}}{Dt} = \int_{c.v.} \eta\rho\,\hat{\mathbf{n}}\cdot\mathbf{V}\,dA$
- **Often one inlet (A_1), and one outlet (A_2)**: $\frac{DN_{sys}}{Dt} = \int_{A_2} \eta_2\rho_2 V_2\,dA - \int_{A_1} \eta_1\rho_1 V_1\,dA$



- **For uniform properties over a plane area**: $\frac{DN_{sys}}{Dt} = \eta_2\rho_2 V_2 A_2 - \eta_1\rho_1 V_1 A_1$

4.3 System-to-Control-Volume Transformation

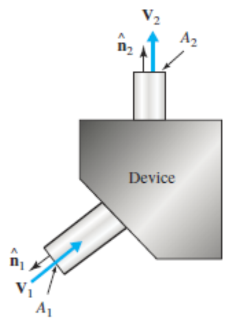
4.3.1 Simplifications of the Reynolds Transport Theorem

$$\frac{DN_{sys}}{Dt} = \frac{d}{dt}\int_{c.v.} \eta\rho \, d\mathcal{V} + \int_{c.s.} \eta\rho\hat{\mathbf{n}}\cdot\mathbf{V}\, dA$$

$$\frac{DN_{sys}}{Dt} = \int_{c.v.} \frac{\partial}{\partial t}(\rho\eta)\, d\mathcal{V} + \int_{c.s.} \eta\rho\hat{\mathbf{n}}\cdot\mathbf{V}\, dA$$

- **Unsteady flow with uniform flow properties**:

$$\frac{DN_{sys}}{Dt} = \mathcal{V}_{c.v.}\frac{d(\eta\rho)}{dt} + \eta_2\rho_2 V_2 A_2 - \eta_1\rho_1 V_1 A_1$$



4.4 Conservation of Mass

$$\frac{Dm_{\text{sys}}}{Dt} = \frac{D}{Dt}\int_{\text{sys}} \rho \, d\tilde{V} = 0$$

Mass of a system is fixed.

- For a steady flow, this simplifies to:

$$\int_{\text{c.s.}} \rho \hat{\mathbf{n}} \cdot \mathbf{V} \, dA = 0$$

- Uniform flow with one entrance and one exit:

$$\rho_2 A_2 V_2 = \rho_1 A_1 V_1$$

For constant density, the continuity equation is only dependent on A and V

4.4 Conservation of Mass

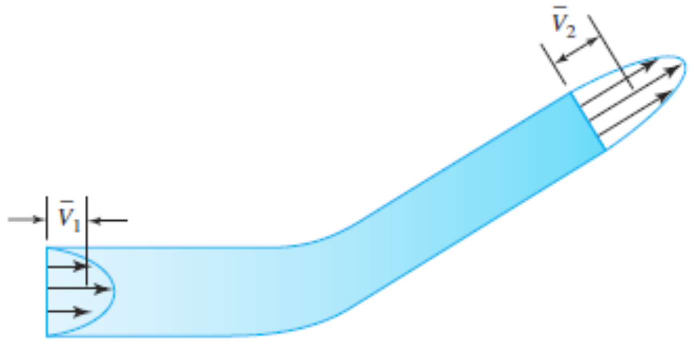


Figure 4.7 Nonuniform velocity profiles.

- If the density is uniform over each area, with nonuniform velocity profiles:

$$\rho_1 \int_{A_1} V_1\, dA = \rho_2 \int_{A_2} V_2\, dA \qquad \rho_1 \overline{V}_1 A_1 = \rho_2 \overline{V}_2 A_2$$ (averages can also be used)

- The *mass flux* $\dot{m}$ (kg/s) is the mass rate of flow: $\dot{m} = \int_A \rho V_n\, dA$
  - Where V_n is the normal component of velocity.

4.4 Conservation of Mass

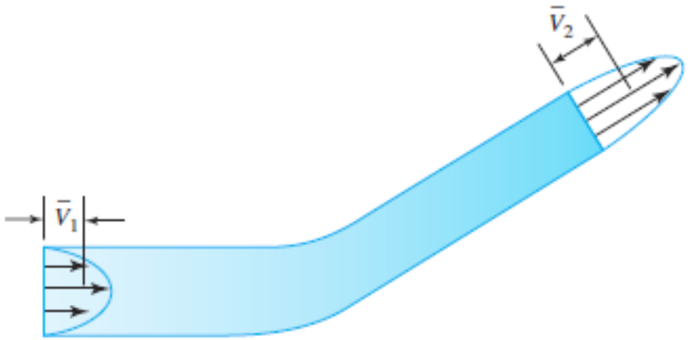


Figure 4.7 Nonuniform velocity profiles.

- The *flow rate (or discharge)* Q (m^3/s) is the volume rate of flow: $Q = \int_A V_n \, dA$
  - Mass flow rate is often used in compressible flow. The flow rate is often used to specify incompressible flow.

4.4 Conservation of Mass

Water flows at a uniform velocity of 3 m/s into a nozzle that reduces the diameter from 100 mm to 20 mm (Figure E4.1). Calculate the water's velocity leaving the nozzle and the flow rate.

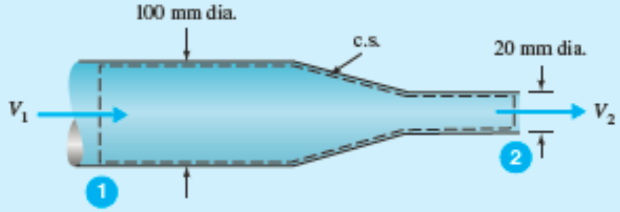


Figure E4.1

Solution

The control volume is selected to be the inside of the nozzle as shown. Flow enters the control volume at section 1 and leaves at section 2. The simplified continuity equation (4.4.6) is used since the density of water is assumed constant and the velocity profiles are uniform:

$$A_1V_1 = A_2V_2$$

$$\therefore V_2 = V_1\frac{A_1}{A_2} = 3\,\frac{\pi \times 0.1^2/4}{\pi \times 0.02^2/4} = \underline{75\text{ m/s}}$$

The flow rate, or discharge, is found to be

$$Q = V_1A_1$$
$$= 3 \times \pi \times 0.1^2/4 = \underline{0.0236\text{ m}^3/\text{s}}$$

4.4 Conservation of Mass

Water flows in and out of a device as shown in Figure E4.2a. Calculate the rate of change of the mass of water (dm/dt) in the device.

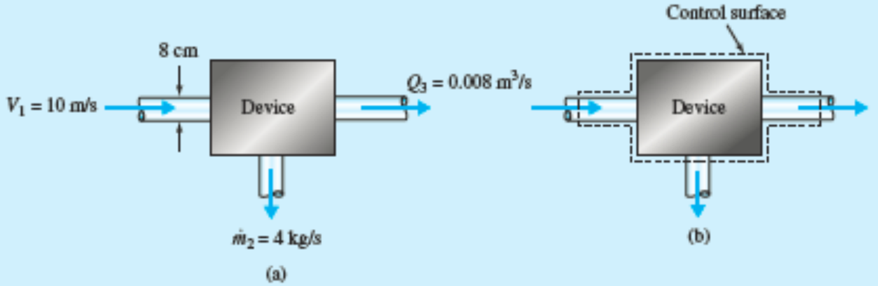


Figure E4.2

Solution

The control surface of the control volume selected is shown in Figure E4.2b. The continuity equation (4.4.2), with three surfaces across which water flows, takes the following form:

$$0 = \frac{d}{dt}\int_{\text{c.v.}} \rho\, d\forall + \int_{\text{c.s.}} \rho \hat{\mathbf{n}} \cdot \mathbf{V}\, dA$$
$$= \frac{dm}{dt} - \rho_1 A_1 V_1 + \rho_2 A_2 V_2 + \rho_3 A_3 V_3$$

where we have assumed the density to be constant over the volume and we have used $\mathbf{V}_1 \cdot \hat{\mathbf{n}} = -V_1$, since $\hat{\mathbf{n}}_1$ points out of the volume, opposite to the direction of $\mathbf{V}_1$. The last three terms come from the area integral. In terms of the quantities given, the above can be expressed as

$$0 = \frac{dm}{dt} - \rho_1 A_1 V_1 + \dot{m}_2 + \rho_3 Q_3$$
$$= \frac{dm}{dt} - 1000\ \text{kg/m}^3 \times \left(\pi \times \frac{0.04^2}{10\,000}\right) \text{m}^2 \times 10\ \text{m/s} + 4\ \text{kg/s}$$
$$+ 1000\ \text{kg/m}^3 \times (0.008)\ \text{m}^3/\text{s}$$

This is solved to yield

$$\frac{dm}{dt} = \underline{38.3\ \text{kg/s}}$$

Hence the mass is increasing at the rate of 38.3 kg/s. To accomplish this, the device could contain a spongelike material that absorbs water.

4.4 Conservation of Mass

A uniform flow of air approaches a cylinder as shown in Figure E4.3a. The symmetrical velocity distribution at the location shown downstream in the wake of the cylinder is approximated by

$$u(y) = 1.25 + \frac{y^2}{4} \qquad -1 < y < 1$$

where $u(y)$ is in m/s and y is in meters. Determine the mass flux across the surface AB per meter of depth (into the paper). Use $\rho = 1.23$ kg/m³.

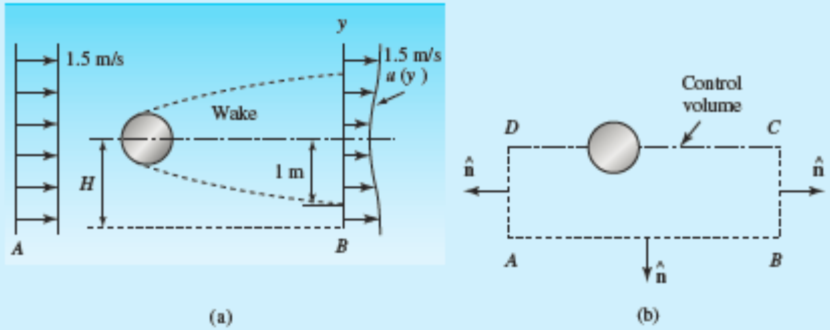


Figure E4.3

Solution

Select $ABCD$ as the control volume (Figure E4.3b). Outside the wake (a region of retarded flow) the velocity is constant at 1.5 m/s. Hence the velocity normal to plane AD is 1.5 m/s. No mass flux crosses the surface CD because of symmetry. Assuming a steady flow, the continuity equation (4.4.3) becomes

$$0 = \int_{c.s.} \rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA$$

Mass flux occurs across three surfaces: AB, BC, and AD. Thus the equation above takes the form

$$0 = \int_{A_{AB}} \rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA + \int_{A_{BC}} \rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA + \int_{A_{AD}} \rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA$$

$$= \dot{m}_{AB} + \int_0^H \rho u(y)\, 1 \times dy - \rho \text{ kg/m}^3 \times 1.5 \text{ m/s} \times H \text{ m} \times 1 \text{ m}$$

where the negative sign for surface AD results from the fact that the unit vector points out of the volume to the left while the velocity vector points to the right. Recall that a negative sign in the steady-flow continuity equation is always associated with an influx and a positive sign with an outflux. Now, we integrate out to 1 m instead of H, since the mass that enters on the left beyond 1 m simply leaves on the right with no net gain or loss. So, letting $H = 1$ m, we have

$$0 = \dot{m}_{AB} + \int_0^1 1.23\left(1.25 + \frac{y^2}{4}\right) dy - (1.23 \times 1 \times 1.5)$$

Perform the integration and there results

$$\dot{m}_{AB} = \underline{0.205 \text{ kg/s per meter}}$$

4.4 Conservation of Mass

A balloon is being inflated with a water supply of 0.6 m^3/s (Figure E4.4a). Find the rate of growth of the radius at the instant when $R = 0.5$ m.

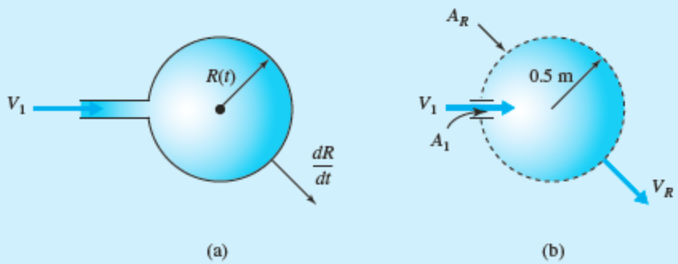


Figure E4.4

Solution

The objective is to find dR/dt when the radius $R = 0.5$ m. This growth rate $V_R = dR/dt$ is the same as the water velocity normal to the wall of the balloon. Therefore, we select as our fixed control volume a sphere with a constant radius of 0.5 m (see Figure E4.4b) so that we can calculate the velocity of the water at the surface at the instant shown moving radially out at $R = 0.5$ m. The continuity equation is written as

$$0 = \int_{c.v.} \underbrace{\frac{\partial \rho}{\partial t}}_{0} dV + \int_{c.s.} \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA$$

The first term is zero because the density of water inside the control volume does not change in time. Further, the water crosses two areas: the inlet area A_1 with a velocity V_1 and the remainder of the sphere surface A_R with a velocity V_R. We will assume that $A_1 << A_R$. The continuity equation then takes the form

$$0 = -\rho A_1 V_1 + \rho A_R V_R$$

Since the flow rate into the volume is $A_1 V_1 = 0.6\ m^3/s$ and $A_R \simeq 4\pi R^2$ assuming that A_1 is quite small, we can solve for V_R. At $R = 0.5$ m

$$V_R = \frac{A_1 V_1}{4\pi R^2} = \frac{0.6\ m^3/s}{4\pi \times 0.5^2\ m^2} = 0.191\ m/s$$

$$\therefore \frac{dR}{dt} = \underline{0.191\ m/s}$$

We have used a fixed control volume and allowed the moving surface of the balloon to pass through it at the instant considered. With this approach it is possible to model situations in which surfaces, such as a piston, are allowed to move.

4.4 Conservation of Mass

This example shows that there may be more than one good choice for a control volume. We want to determine the rate at which the water level rises in an open container if the water coming in through a 0.10-m^2 pipe has a velocity of 0.5 m/s and the flow rate going out is 0.2 m^3/s (Figure E4.5a). The container has a circular cross section with a diameter of 0.5 m.

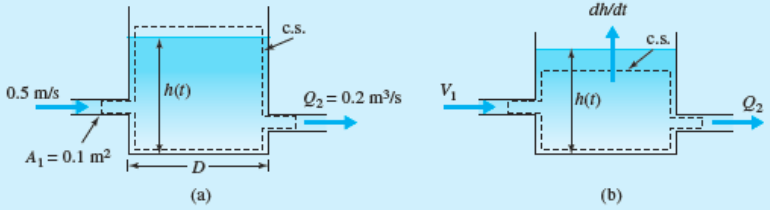


Figure E4.5

Solution

First we select a control volume that extends above the water surface as shown in Figure E4.5a. Apply the continuity equation (Eq. 4.4.2):

$$\frac{d}{dt}\int_{\text{c.v.}} \rho\, d\forall + \rho(-V_1)A_1 + \rho V_2 A_2 = 0$$

in which the first term describes the rate of change of mass in the control volume. Hence, neglecting the airmass above the water, we have

$$\frac{d(\rho h \pi D^2/4)}{dt} - \rho V_1 A_1 + \rho Q_2 = 0$$

Divide by the constant ρ,

$$\frac{\pi D^2}{4}\frac{dh}{dt} - V_1 A_1 + Q_2 = 0$$

The rate at which the water level is rising is then

$$\frac{dh}{dt} = \frac{V_1 A_1 - Q_2}{\pi D^2/4}$$

Thus

$$\frac{dh}{dt} = \frac{(0.5 \times 0.1 - 0.2)\ \text{m}^3/\text{s}}{(\pi \times 0.5^2/4)\ \text{m}^2} = \underline{-0.764\ \text{m/s}}$$

The negative sign indicates that the water level is actually decreasing.

Let's solve this problem again but with another choice for the control volume, one with its top surface below the water level (Figure E4.5b). The velocity at the top surface is then equal to the rate at which the surface rises, i.e., dh/dt. The flow condition inside the control volume is steady. Hence we can apply Eq. 4.4.4. There are three areas across which fluid flows. On the third area, the velocity is dh/dt; hence the continuity equation takes the form

$$\rho(-V_1)A_1 + \rho Q_2 + \rho\frac{dh}{dt}\frac{\pi}{4}D^2 = 0$$

so that

$$\frac{dh}{dt} = \frac{V_1 A_1 - Q_2}{\pi D^2/4}$$

This is the same result as given above.

4.5 Energy Equation

- This equation is required if heat is transferred (boiler/compressor) or work is done (pump/turbine).
  - Can relate pressures/velocities when Bernoulli's equation cannot be used.

$$\dot{Q} - \dot{W} = \frac{D}{Dt}\int_{\text{sys}} e\rho \, d\forall$$

Where e is the specific energy and consists of the specific kinetic energy, specific potential energy, and specific internal energy.

$$e = \frac{V^2}{2} + gz + \overline{u}$$

- In terms of a control volume: $\dot{Q} - \dot{W} = \frac{d}{dt}\int_{\text{c.v.}} e\rho \, d\forall + \int_{\text{c.s.}} \rho e \mathbf{V} \cdot \hat{\mathbf{n}} \, dA$

4.5 Energy Equation

- In terms of a control volume: $\dot{Q} - \dot{W} = \frac{d}{dt}\int_{\text{c.v.}} e\rho \, d\forall + \int_{\text{c.s.}} \rho e \mathbf{V} \cdot \hat{\mathbf{n}} \, dA$
- $\dot{Q}$: Rate-of-energy transfer across the control surface due to a temperature difference.
- $\dot{W}$: Work-rate term due to work being done by the system.

4.5 Energy Equation

4.5.1 Work-Rate Term

- The work-rate term is from the work being done by the system.
- Rate of work (Power) is the dot product of force with its velocity.

$$\dot{W} = -\mathbf{F} \cdot \mathbf{V}_I$$

The velocity is measured w.r.t. a fixed inertial reference frame. Negative sign is because work done on the control volume is negative.

- If the force is from variable stress over a control surface:

$$\dot{W} = -\int_{\text{c.s.}} \boldsymbol{\tau} \cdot \mathbf{V}_I \, dA$$

  - τ is a stress vector acting on an elemental area dA [A differential force].

4.5 Energy Equation

4.5.1 Work-Rate Term

$$\dot{W} = \int_{\text{c.s.}} p\hat{\mathbf{n}} \cdot \mathbf{V}\, dA + \dot{W}_S + \dot{W}_{\text{shear}} + \dot{W}_I$$

- $\int p\hat{\mathbf{n}} \cdot \mathbf{V}\, dA$ Work rate resulting from the force due to pressure moving at the control surface. It is often referred to as **flow work**.
- $\dot{W}_S$ Work rate resulting from rotating shafts such as that of a pump or turbine, or the equivalent electric power.
- $\dot{W}_{\text{shear}}$ Work rate due to the shear acting on a moving boundary, such as a moving belt.
- $\dot{W}_I$ Work rate that occurs when the control volume moves relative to a fixed reference frame.

4.5 Energy Equation

4.5.2 General Energy Equation

- From the previous equation, the energy equation can be rewritten as:

$$\dot{Q} - \dot{W}_S - \dot{W}_{\text{shear}} - \dot{W}_I = \frac{d}{dt}\int_{\text{c.v.}} e\rho \, dV\!\!\!/ + \int_{\text{c.s.}} \left(e + \frac{p}{\rho}\right) \rho \hat{\mathbf{n}} \cdot \mathbf{V} \, dA$$

- Losses are the sum of all terms for unusable forms of energy.

$$\text{losses} = -\dot{Q} + \frac{d}{dt}\int_{\text{c.v.}} \tilde{u} \, \rho \, dV + \int_{\text{c.s.}} \tilde{u}\rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA$$

  - Can be due to viscosity (causes friction resulting in increased internal energy).
  - Or due to changes in geometry resulting in separated flows.

4.5 Energy Equation

4.5.3 Steady Uniform Flow

- For steady-flow with one inlet and one outlet (with uniform profile) and no shear work, the following energy equation is used:

$$-\frac{\dot{W}_S}{\dot{m}g} = \frac{V_2^2 - V_1^2}{2g} + \frac{p_2}{\gamma_2} - \frac{p_1}{\gamma_1} + z_2 - z_1 + h_L$$

- Where h_L is the head loss (dimensions of length).

$$h_L = \frac{\tilde{u}_2 - \tilde{u}_1}{g} - \frac{\dot{Q}}{\dot{m}g}$$

$$h_L = K\frac{V^2}{2g}$$

Where **K** is the loss coefficient

- $\frac{V^2}{g}$ is the velocity head, and $\frac{p}{\gamma}$ is the pressure head.

4.5 Energy Equation

4.5.3 Steady Uniform Flow

- For steady-flow with one inlet and one outlet (with uniform profiles) and no shear work, negligible losses, and no shaft work:

$$\frac{V_2^2}{2g} + \frac{p_2}{\gamma_2} + z_2 = \frac{V_1^2}{2g} + \frac{p_1}{\gamma_1} + z_1$$

Almost identical to Bernoulli's equation for a constant density flow.

- The pump head, H_P is the energy term associated for a pump $[\frac{\dot{W}_S}{\dot{m}g}]$. If a turbine is involved, the energy term is called the turbine head.

4.5 Energy Equation

4.5.3 Steady Uniform Flow

- If a turbine/pump is used, the efficiency of a device is needed, η_T
  - The power generated by the turbine is:

$$\dot{W}_T = \dot{m} g H_T \eta_T = \gamma Q H_T \eta_T$$

$$\dot{m} = \rho A V$$

  - The power required by a pump is:

The power is calculated in Watts

$$\dot{W}_P = \frac{\dot{m} g H_P}{\eta_P} = \frac{\gamma Q H_P}{\eta_P}$$

4.5 Energy Equation

4.5.4 Steady Nonuniform Flow

- If a uniform velocity profile assumption cannot be used, the velocity distribution should be corrected:
  - Using a **kinetic-energy correction factor α**

$$\alpha = \frac{\int V^3 dA}{\bar{V}^3 A}$$

  - The term that accounts for the flux in kinetic energy is:

$$\frac{1}{2}\rho\int_A V^3\, dA = \frac{1}{2}\alpha\rho\bar{V}^3 A$$

With $\bar{V}$ being the average velocity over area A

- The final equation that account for this nonuniform velocity distribution is:

$$H_P + \alpha_1\frac{\bar{V}_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = H_T + \alpha_2\frac{\bar{V}_2^2}{2g} + \frac{p_2}{\gamma} + z_2 + h_L$$

4.5 Energy Equation

The pump of Figure E4.6 is to increase the pressure of 0.2 m^3/s of water from 200 kPa to 600 kPa. If the pump is 85% efficient, how much electrical power will the pump require? The exit area is 20 cm above the inlet area. Assume inlet and exit areas are equal.

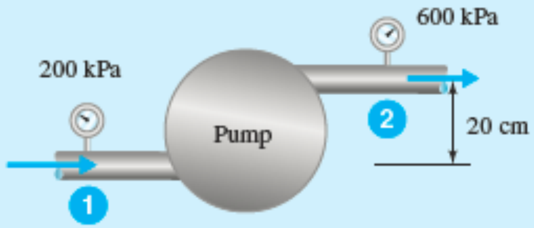


Figure E4.6

Solution

The energy equation (4.5.24) across the pump provides the energy delivered to the water as a pump head:

$$H_P = \frac{p_2 - p_1}{\gamma} + z_2 - z_1$$

$$= \frac{(600\,000 - 200\,000)\,\text{N/m}^2}{9810\ \text{N/m}^3} + 0.2\ \text{m} = 41.0\ \text{m}$$

where $V_2 = V_1$ since the inlet and exit areas are equal, and any losses are accounted for with the efficiency of Eq. 4.5.26. That equation provides the power required by the pump:

$$\dot{W}_P = \frac{\gamma Q H_P}{\eta_P}$$

$$= \frac{9810\ \text{N/m}^3 \times 0.2\ \text{m}^3/\text{s} \times 41.0\ \text{m}}{0.85} = 94\,600\ \text{J/s} \quad \text{or} \quad \underline{94.6\ \text{kW}}$$

4.5 Energy Equation

Water flows from a reservoir through a 800-mm-diameter pipeline to a turbine-generator unit and exits to a river that is 30 m below the reservoir surface. If the flow rate is 3 m^3/s, and the turbine-generator efficiency is 88%, calculate the power output. Assume the loss coefficient in the pipeline (including the exit) to be $K = 2$.

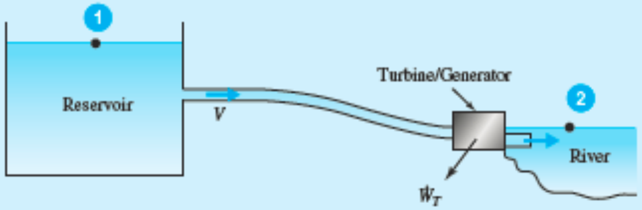


Figure E4.7

Solution

Referring to Figure E4.7, we select the control volume to extend from section 1 to section 2 on the reservoir and river surfaces, where we know the velocities, pressures, and elevations; we consider the water surface of the left reservoir to be the entrance and the water surface of the river to be the exit. The velocity in the pipe is

$$V = \frac{Q}{A} = \frac{3}{\pi \times 0.8^2/4} = 5.97 \text{ m/s}$$

Now, consider the energy equation. We will use gage pressures so that $p_1 = p_2 = 0$; the datum is placed through the lower section 2 so that $z_2 = 0$; the velocities V_1 and V_2 on the reservoir surfaces are negligibly small; K is assumed to be based on the 800 mm-diameter pipe velocity. The energy equation (4.5.24) then becomes

$$\overset{0}{\cancel{H_P}} + \overset{0}{\cancel{\frac{V_1^2}{2g}}} + \overset{0}{\cancel{\frac{p_1}{\gamma}}} + z_1 = H_T + \overset{0}{\cancel{\frac{V_2^2}{2g}}} + \overset{0}{\cancel{\frac{p_2}{\gamma}}} + \overset{0}{\cancel{z_2}} + K\frac{V^2}{2g}$$

$$30 = H_T + 2\frac{5.97^2 \text{ m}^2/\text{s}^2}{2 \times 9.81 \text{ m/s}^2}$$

$$\therefore H_T = 26.4 \text{ m}$$

From this the power output is found using Eq. 4.5.25 to be

$$\dot{W}_T = 3 \text{ m/s}^2 \times 9810 \text{ N/m}^3 \times 26.4 \text{ m} \times 0.88 = \underline{684 \text{ kW}}$$

In this example we have used gage pressure; the potential-energy datum was assumed to be placed through section 2, V_1 and V_2 were assumed to be insignificantly small, and K was assumed to be based on the 762-mm-diameter pipe velocity.

4.5 Energy Equation

The venturi meter shown reduces the pipe diameter from 100 mm to a minimum of 50 mm (Figure E4.8). Calculate the flow rate and the mass flux assuming ideal conditions.

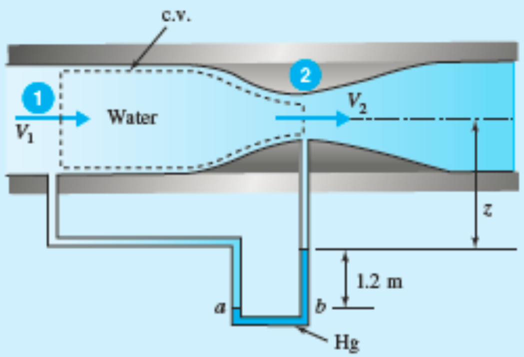


Figure E4.8

Solution

The control volume is selected as shown such that the entrance and exit correspond to the sections where the pressure information of the manometer can be applied. The manometer's reading is interpreted as follows:

$$p_a = p_b$$
$$p_1 + \gamma(z + 1.2) = p_2 + \gamma z + 13.6\gamma \times 1.2$$

where z is the distance from the pipe centerline to the top of the mercury column. The manometer then gives

$$\frac{p_1 - p_2}{\gamma} = (13.6 - 1) \times 1.2 = 15.12\,\text{m}$$

4.5 Energy Equation

Continuity (4.4.6) allows us to relate V_2 to V_1 by

$$V_1 A_1 = V_2 A_2$$

$$\therefore \; V_2 = \frac{A_1}{A_2} V_1 = \frac{\pi \times 10^3/4}{\pi \times 5^2/4} V_1 = 4V_1$$

The energy equation (4.5.17) assuming ideal conditions (no losses and uniform flow) with $h_L = \dot{W}_S = 0$ takes the form

$$0 = \frac{V_2^2 - V_1^2}{2g} + \frac{p_2 - p_1}{\gamma} + \cancelto{0}{(z_2 - z_1)}$$

$$= \frac{16V_1^2 - V_1^2}{2g} - 15.12$$

$$\therefore \; V_1 = 4.45 \text{m/s}$$

The flow rate is

$$Q = A_1 V_1 = (\pi \times 0.05^2) \times 4.45 = \underline{0.0350 \text{ m}^3/\text{s}}$$

The mass flux is

$$\dot{m} = \rho Q = 1000 \times 0.035 = \underline{35.0 \text{ kg/s}}$$

4.5 Energy Equation

The velocity distribution for a certain flow in a pipe is $V(r) = V_{\text{max}}(1 - r^2/r_0^2)$, where r_0 is the pipe radius (Figure E4.9). Determine the kinetic-energy correction factor.

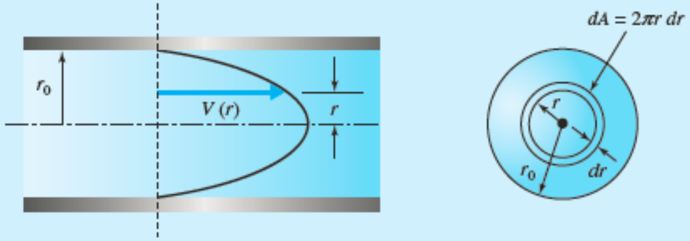


Figure E4.9

Solution

To find the kinetic-energy correction factor α, we must know the average velocity. It is (combine Eqs. 4.4.10 and 4.4.11)

$$\bar{V} = \frac{\int V dA}{A}$$

$$= \frac{1}{\pi r_0^2}\int_0^{r_0} V_{\text{max}}\left(1 - \frac{r^2}{r_0^2}\right)2\pi r\,dr \quad = \frac{2\pi V_{\text{max}}}{\pi r_0^2}\int_0^{r_0}\left(r - \frac{r^3}{r_0^2}\right)dr$$

$$= \frac{2V_{\text{max}}}{r_0^2}\left(\frac{r_0^2}{2} - \frac{r_0^4}{4r_0^2}\right) = \frac{1}{2}V_{\text{max}}$$

Using Eq. 4.5.27, there results

$$\alpha = \frac{\int V^3 dA}{\bar{V}^3 A}$$

$$= \frac{\int_0^{r_0} V_{\text{max}}^3\,(1 - r^2/r_0^2)^3 2\pi r\,dr}{\left(\frac{1}{2}V_{\text{max}}\right)^3 \pi r_0^2} \quad = \frac{16}{r_0^2}\int_0^{r_0}\left(1 - \frac{3r^2}{r_0^2} + \frac{3r^4}{r_0^4} - \frac{r^6}{r_0^6}\right) r\,dr$$

$$= \frac{16}{r_0^2}\left(\frac{r_0^2}{2} - \frac{3r_0^2}{4} + \frac{3r_0^2}{6} - \frac{r_0^2}{8}\right) = 2$$

Consequently, the kinetic energy flux associated with a parabolic velocity distribution across a circular area is given by

$$\int \rho \mathbf{V}\cdot\hat{\mathbf{n}}\frac{V^2}{2}dA = 2 \times \frac{\dot{m}\bar{V}^2}{2} = \dot{m}\bar{V}^2$$

Parabolic velocity distributions are encountered in laminar flows in pipes and between parallel plates, downstream of inlets and geometry changes (valves, elbows, etc.). The Reynolds number must be quite small, usually less than about 2000.

4.5 Energy Equation

The drag force on an automobile (Figure E4.10) is approximated by the expression $0.15\rho V_\infty^2 A$, where A is the projected cross-sectional area and V_∞ is the automobile's speed. If $A = 1.2\ \text{m}^2$, calculate the efficiency η of the engine if the rate of fuel consumption $\dot{f}$ (the gas mileage) is 15×10^3 km/m³ and the automobile travels at 90 km/h. Assume that the fuel releases 44 000 kJ/kg during combustion. Neglect the energy lost due to the exhaust gases and coolant and assume that the only resistance to motion is the drag force. Use $\rho_{\text{air}} = 1.12\ \text{kg/m}^3$ and $\rho_{\text{fuel}} = 680\ \text{kg/m}^3$.

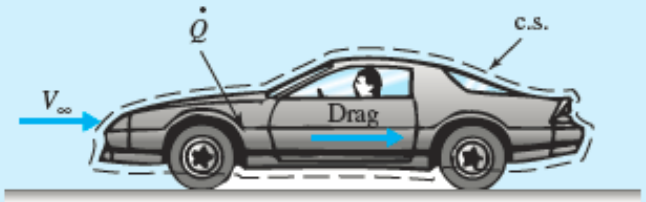


Figure E4.10

Solution

If the car is taken as the moving control volume (note that the control volume is fixed), as shown, we can simplify the energy equation (Eq. 4.5.3 in combination with 4.5.11) to

$$\dot{Q} - \dot{W}_I = 0$$

4.5 Energy Equation

since all other terms are negligible; there is no velocity crossing the control volume, so $\mathbf{V} \cdot \hat{\mathbf{n}} = 0$ (neglect the energy of the exhaust gases); there is no shear or shaft work; the energy of the c.v. remains constant. The energy input $\dot{Q}$ which accomplishes useful work is η times the energy released during combustion; that is,

$$\dot{Q} = \dot{m}_f \times 44\,000\eta \text{ kJ/s}$$

where $\dot{m}_f$ is the mass flux of the fuel. The mass flux of fuel is determined knowing the rate of fuel consumption $\dot{f}$ and the density of fuel as 680 kg/m³, as follows:

$$\dot{f} = \frac{\text{distance}}{\text{volume}} = \frac{V_\infty \times \text{time}}{Q \times \text{time}} = \frac{V_\infty}{\dot{m}_f/\rho_f} = \frac{\rho_f V_\infty}{\dot{m}_f}$$

with $V_\infty = 90\,000/3600 = 25$ m/s, we have, using $\dot{f} = 15 \times 10^6$ m/m³,

$$15 \times 10^6 = \frac{680 \times 25}{\dot{m}_f}$$

$$\therefore \dot{m}_f = 0.001133 \text{ kg/s}$$

The inertial work-rate term is

$$\begin{aligned} \dot{W}_I &= V_\infty \times \text{drag} \\ &= 0.15\rho V_\infty^3 A = 0.15 \times 1.12 \times 25^3 \times 1.2 = 3150 \text{ J/s} \end{aligned}$$

Equating $\dot{Q} = \dot{W}_I$, we have

$$44\,000\eta \times 0.001133 = 3.15$$

$$\therefore \eta = 0.0632 \quad \text{or} \quad \underline{6.32\%}$$

This is obviously a very low percentage, perhaps surprisingly low to the reader. Very little power (3.15 kJ/s = 4.22 hp) is actually needed to propel the automobile at 90 km/h. The relatively large engine, needed primarily for acceleration, is quite inefficient when simply propelling the automobile.

Note the importance of using a stationary reference frame. The reference frame attached to the automobile is an inertial reference frame since it is moving at constant velocity. Yet the energy equation demands a stationary reference frame allowing the energy required by the drag force to be properly included.

4.6 Momentum Equation

4.6.1 General Momentum Equation

- Newton's second law (momentum equation): The resultant force acting on a system equals the rate of change of momentum of the system in an inertial reference frame.

$$\Sigma \mathbf{F} = \frac{D}{Dt}\int_{\text{sys}} \rho \mathbf{V}\, d\forall$$

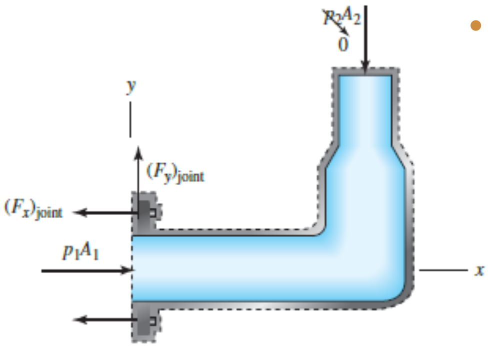
- For a control volume: $$\Sigma \mathbf{F} = \frac{d}{dt}\int_{\text{c.v.}} \rho \mathbf{V}\, d\forall + \int_{\text{c.s.}} \rho \mathbf{V}(\mathbf{V}\cdot\hat{\mathbf{n}})\, dA$$

4.6 Momentum Equation

4.6.2 Steady Uniform Flow

- If flow is uniform and steady, for N number of entrances and exits, the previous equation can be simplified to:

$$\Sigma \mathbf{F} = \sum_{i=1}^{N} \rho_i A_i \mathbf{V}_i (\mathbf{V}_i \cdot \hat{\mathbf{n}})$$



Horizontal nozzle with one entrance and one exit

- The momentum equation simplifies to:

$$\Sigma \mathbf{F} = \rho_2 A_2 V_2 \mathbf{V}_2 - \rho_1 A_1 V_1 \mathbf{V}_1$$

With continuity: $\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2$

$$\Sigma \mathbf{F} = \dot{m}(\mathbf{V}_2 - \mathbf{V}_1)$$

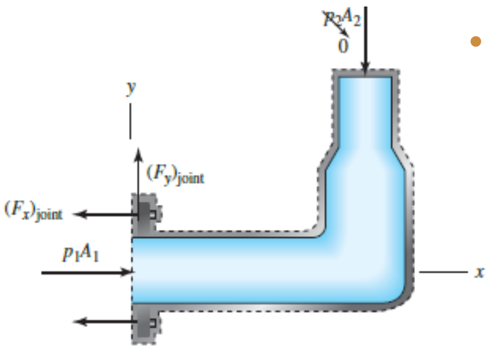
$$\Sigma F_x = \dot{m}(V_{2x} - V_{1x})$$

$$\Sigma F_y = \dot{m}(V_{2y} - V_{1y})$$

$$\Sigma F_z = \dot{m}(V_{2z} - V_{1z})$$

4.6 Momentum Equation

4.6.2 Steady Uniform Flow



Horizontal nozzle with one entrance and one exit

- To determine the x-component of the force of the joint on the nozzle:

$$\Sigma \mathbf{F} = \dot{m}(\mathbf{V}_2 - \mathbf{V}_1)$$

$$\Sigma F_x = -(F_x)_{\text{joint}} + p_1 A_1 = -\dot{m} V_1$$ As $(V_1)_x = V_1$ and $(V_2)_x = 0$

4.7 Summary

Continuity	*Energy*	*Momentum*
	General Form	
$0 = \frac{d}{dt}\int_{\text{c.v.}} \rho \, d\mathcal{V} + \int_{\text{c.s.}} \rho \mathbf{V}\cdot\hat{\mathbf{n}}\, dA$	$-\Sigma \dot{W} = \frac{d}{dt}\int_{\text{c.v.}} \left(\frac{V^2}{2} + gz\right)\rho \, d\mathcal{V} + \int_{\text{c.s.}} \left(\frac{V^2}{2} + \frac{p}{\rho} + gz\right)\rho \mathbf{V}\cdot\hat{\mathbf{n}}\, dA + \text{losses}$	$\Sigma \mathbf{F} = \frac{d}{dt}\int_{\text{c.v.}} \rho \mathbf{V}\, d\mathcal{V} + \int_{\text{c.s.}} \rho \mathbf{V}(\mathbf{V}\cdot\hat{\mathbf{n}})\, dA$
	Steady Flow	
$0 = \int_{\text{c.s.}} \rho \mathbf{V}\cdot\hat{\mathbf{n}}\, dA$	$-\Sigma \dot{W} = \int_{\text{c.s.}} \left(\frac{V^2}{2} + \frac{p}{\rho} + gz\right)\rho \mathbf{V}\cdot\hat{\mathbf{n}}\, dA + \text{losses}$	$\Sigma \mathbf{F} = \int_{\text{c.s.}} \rho \mathbf{V}(\mathbf{V}\cdot\hat{\mathbf{n}})\, dA$

4.8 Summary

Steady Nonuniform Form[1]

$$\dot{m} = \rho_1 A_1 \bar{V}_1 = \rho_2 A_2 \bar{V}_2$$

$$\frac{-\Sigma \dot{W}}{\dot{m}g} = \alpha_2 \frac{\bar{V}_2^2}{2g} + \frac{p_2}{\gamma_2} + z_2 - \alpha_1 \frac{\bar{V}_1^2}{2g} - \frac{p_1}{\gamma_1} - z_1 + h_L$$

$$\Sigma F_x = \dot{m}(\beta_2 \bar{V}_{2x} - \beta_1 \bar{V}_{1x})$$
$$\Sigma F_y = \dot{m}(\beta_2 \bar{V}_{2y} - \beta_1 \bar{V}_{1y})$$

Steady Uniform Form[1]

$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

$$-\frac{\Sigma \dot{W}}{\dot{m}g} = \frac{V_2^2}{2g} + \frac{p^2}{\gamma^2} + z_2 - \frac{V_1^2}{2g} - \frac{p_1}{\gamma} - z_1 + h_L$$

$$\Sigma \mathbf{F} = \dot{m}(\mathbf{V}_2 - \mathbf{V}_1)$$

Steady Uniform Incompressible Flow[1]

$$Q = A_1 V_1 = A_2 V_2$$

$$-\frac{\Sigma \dot{W}}{\dot{m}g} = \frac{V_2^2}{2g} + \frac{p^2}{\gamma} + z_2 - \frac{V_1^2}{2g} - \frac{p_1}{\gamma} - z_1 + h_L$$

$$\Sigma \mathbf{F} = \dot{m}(\mathbf{V}_2 - \mathbf{V}_1)$$

$$H_p + \frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = H_T + \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2 + h_L$$

$\dot{m}$ = mass flux

Q = flow rate

$\bar{V}$ = average velocity $= \dfrac{\int V dA}{A}$

α = kinetic energy correction factor $= \dfrac{\int V^3 dA}{\bar{V}^3 A}$

β = momentum correction factor $= \dfrac{\int V^2 dA}{\bar{V}^2 A}$

h_L = head loss

$\Sigma \dot{W} = \dot{W}_S + \dot{W}_{shear} + \dot{W}_I$

H_p = pump head = $\dot{W}_p/\dot{m}g$

H_T = turbine head = $\dot{W}_T/\dot{m}g$

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[1]The control volume has one entrance (section 1) and one exit (section 2).