IDEAL FLOW

COORDINATE SYSTEM

COORDINATE SYSTEM

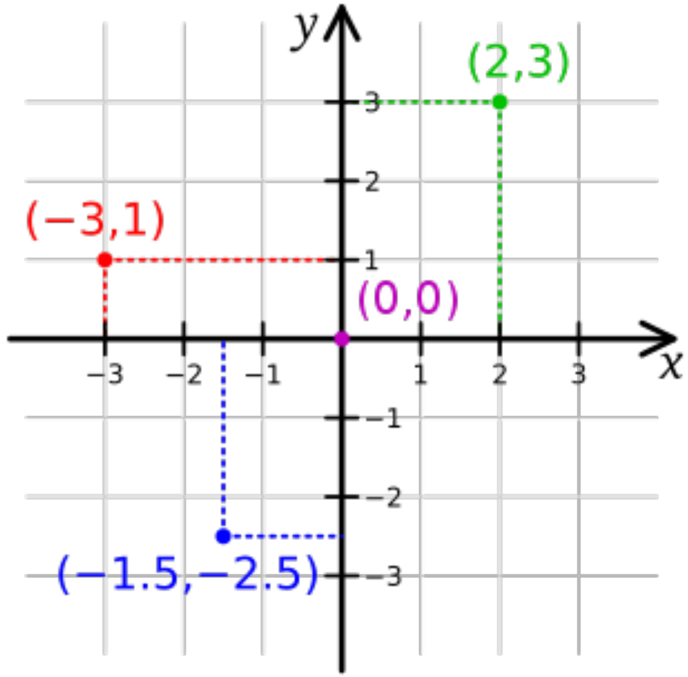
There are two coordinate systems being used in this chapter.

1. Cartesian coordinate

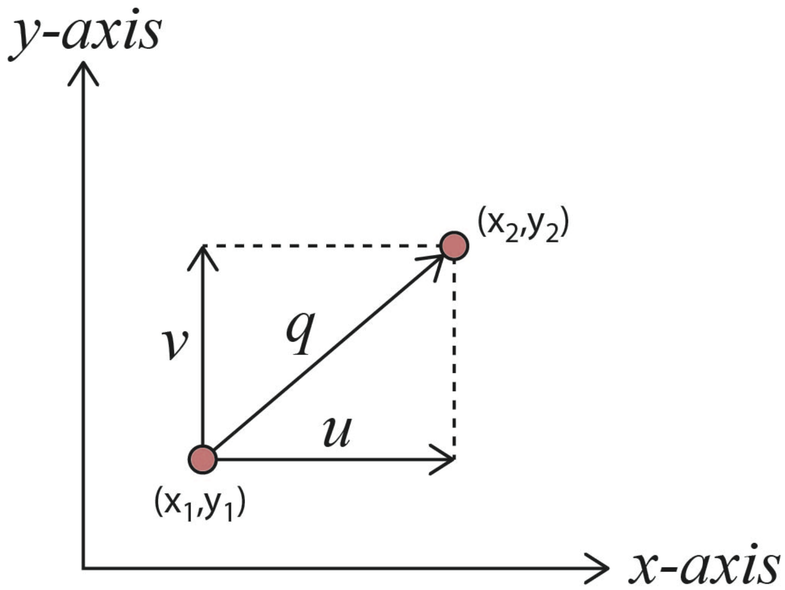
2. Polar coordinate

Cartesian coordinate system:

A Cartesian coordinate system is a coordinate system that specifies each point uniquely in a plane by a pair of numerical coordinates, which are the signed distances to the point from two fixed perpendicular directed lines, measured in the same unit of length. Each reference line is called a coordinate axis or just axis (plural axes) of the system, and the point where they meet is its origin, at ordered pair (0, 0). The coordinates can also be defined as the positions of the perpendicular projections of the point onto the two axes, expressed as signed distances from the origin.



Example of Cartesian coordinate



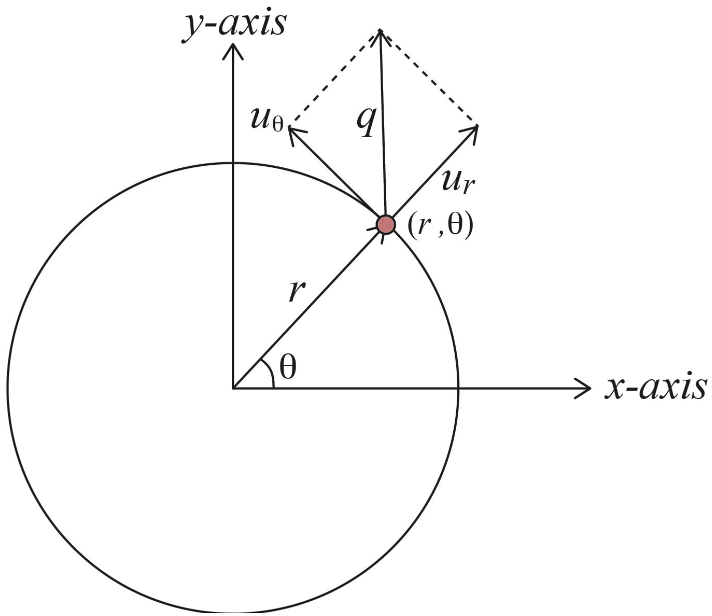
$$u = \text{Velocity on x direction}$$
$$v = \text{Velocity on y direction}$$
$$q = \text{Resultant velocity}$$

$$q = \sqrt{u^2 + v^2}$$

Polar coordinate system:

In mathematics, the polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction.

The reference point (analogous to the origin of a Cartesian coordinate system) is called the pole, and the ray from the pole in the reference direction is the polar axis. The distance from the pole is called the radial coordinate or radius, and the angle is called the angular coordinate, polar angle, or azimuth.



$$u_r = \text{Velocity on radial direction}$$
$$u_\theta = \text{Velocity on angular direction}$$
$$q = \text{Resultant velocity}$$

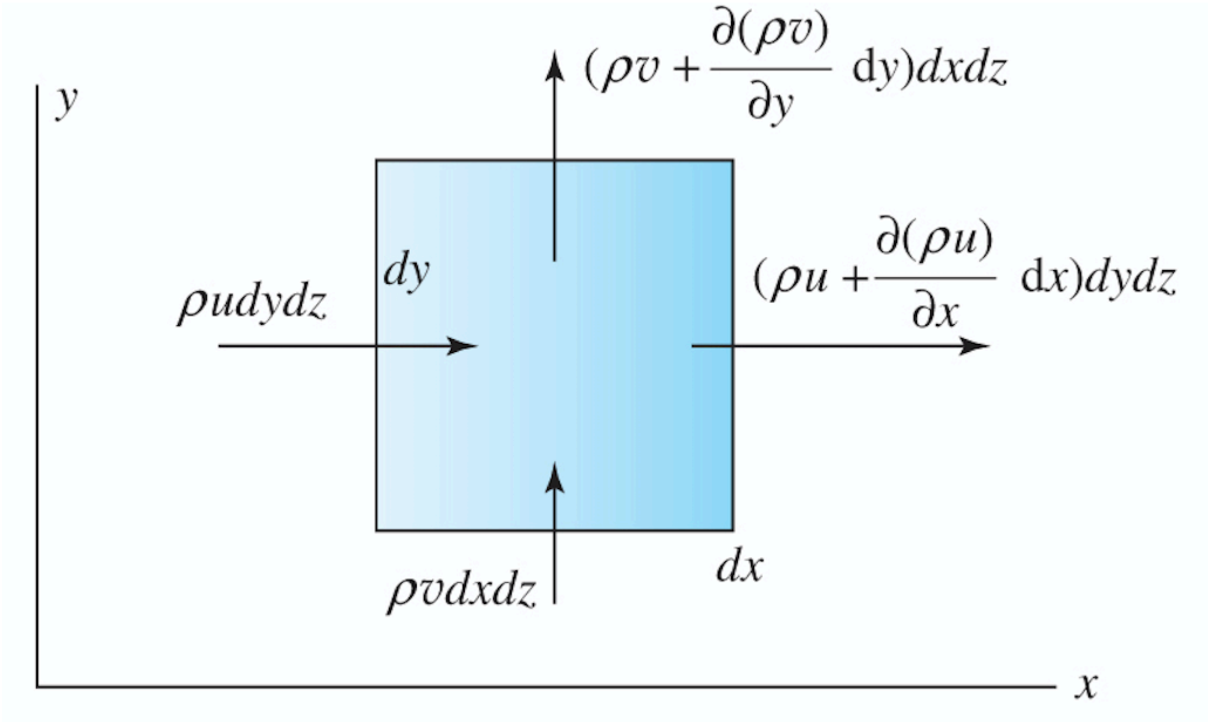
$$q = \sqrt{(u_r)^2 + (u_\theta)^2}$$

IDEAL FLOW

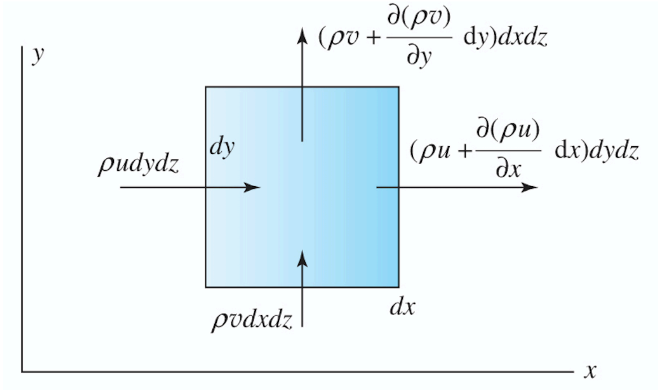
CONTINUITY EQUATION
STREAM FUNCTION

CONTINUITY FLOW IN CARTESIAN CORDINATE

Mass flow rate into the element in x- and y-direction is shown in the figure below.



The net flux of mass entering the element equal to the rate of change of the mass of the element.



$$\dot{m}_{\text{in}} - \dot{m}_{\text{out}} = \frac{\partial}{\partial t} m_{\text{element}}$$

$$\dot{m} = \rho A V \quad , \quad m = \rho \forall \quad (\forall = \text{Volume})$$

$$\rho u dydz + \rho v dxdz - \left(\rho u + \frac{\partial(\rho u)}{\partial x} dx\right) dydz - \left(\rho v + \frac{\partial(\rho v)}{\partial y} dy\right) dxdz = \frac{\partial}{\partial t}(\rho dxdydz)$$

Simplifying the above expression:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial \rho}{\partial t} = 0$$

If the z-direction is exist, it will become:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

Then, the differential continuity equation can be written as:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

This is the most general form of the differential continuity equation expressed using rectangular coordinates.

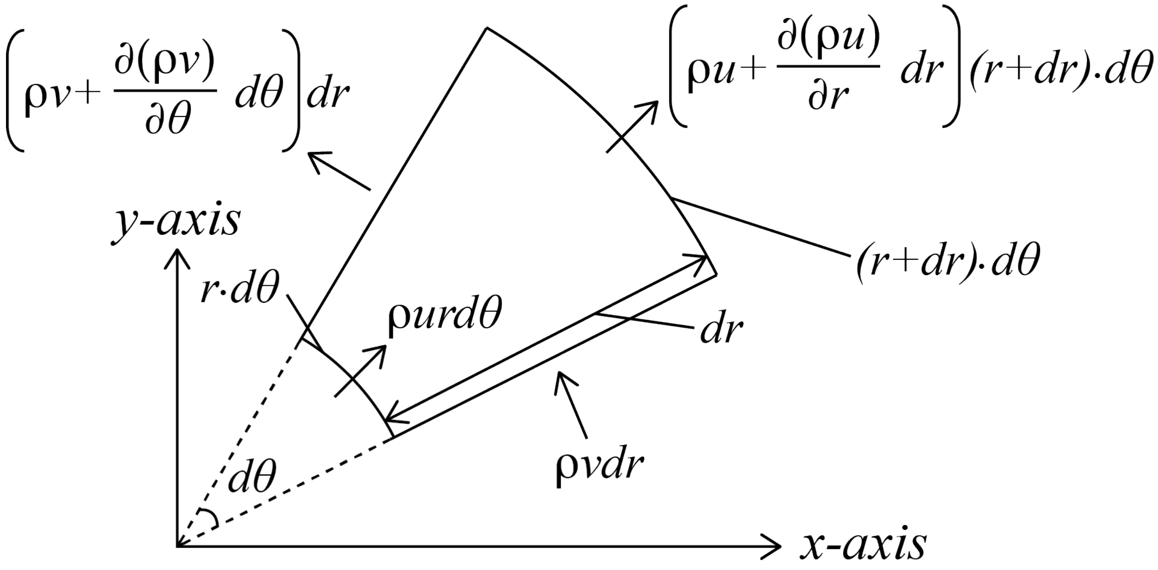
For the case of incompressible flow, a flow in which density of a fluid particle does not change as it travels along, the continuity equation becomes:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Assume that we are discussing only 2-D coordinate, and there is no changes in density (incompressible), we might express the continuity equation as follows:

$$\frac{du}{dx}+\frac{dv}{dy}=0$$

CONTINUITY FLOW IN POLAR CORDINATE



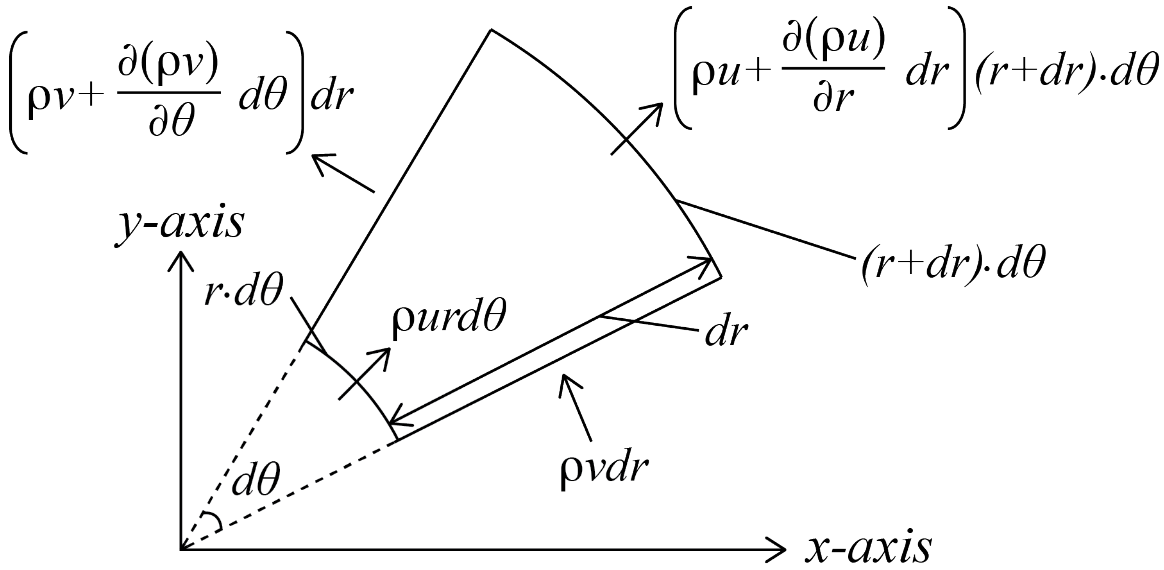
The net flux of mass entering the element equal to the rate of change of the mass of the element. Assume that z-direction is equal to 1 unit.

$$\dot{m}_{\text{in}} - \dot{m}_{\text{out}} = \frac{\partial}{\partial t} m_{\text{element}}$$

$$\dot{m}_{\text{in}} = \rho u r d\theta + \rho v dr$$

$$\dot{m}_{\text{out}} = \left(\rho u + \frac{\partial(\rho u)}{\partial r} dr\right)(r + dr)d\theta + \left(\rho v + \frac{\partial(\rho v)}{\partial \theta} d\theta\right) dr$$

$$\frac{\partial}{\partial t} m_{\text{element}} = \frac{\partial}{\partial t} \rho r d\theta dr$$



the equation can be written as:

$$\rho u r d\theta + \rho v dr - \left(\left(\rho u + \frac{\partial(\rho u)}{\partial r}dr\right)(r + dr)d\theta + \left(\rho v + \frac{\partial(\rho v)}{\partial \theta}d\theta\right)dr\right) = \frac{\partial}{\partial t}\rho r d\theta dr$$

Simplifying the equation.

Assume that $(dr \cdot dr)$ is too small and can be neglected.

$$-\rho u dr d\theta - \left(\frac{\partial(\rho u)}{\partial r}dr d\theta r\right) - \left(\frac{\partial(\rho v)}{\partial \theta}d\theta dr\right) = \frac{\partial}{\partial t}\rho r d\theta dr$$

Divided by $drd\theta r$ will gives :

$$\frac{-\rho u}{r} - \left(\frac{\partial(\rho u)}{\partial r}\right) - \left(\frac{1}{r} \cdot \frac{\partial(\rho v)}{\partial r}\right) = \frac{\partial \rho}{\partial t}$$

$$\frac{\partial \rho}{\partial t} + \frac{\rho u}{r} + \frac{\partial(\rho u)}{\partial r} + \frac{1}{r} \cdot \frac{\partial(\rho v)}{\partial \theta} = 0$$

For incompressible flow, value of density is constant.
The equation will becomes:

$$\frac{u}{r} + \frac{\partial(u)}{\partial r} + \frac{1}{r} \cdot \frac{\partial(v)}{\partial \theta} = 0$$

Officially the continuity equation for incompressible flow in polar coordinate can be written as:

$$\frac{u_r}{r} + \frac{du_r}{dr} + \frac{1}{r} \cdot \frac{du_\theta}{d\theta} = 0$$

STREAM FUNCTION, ψ

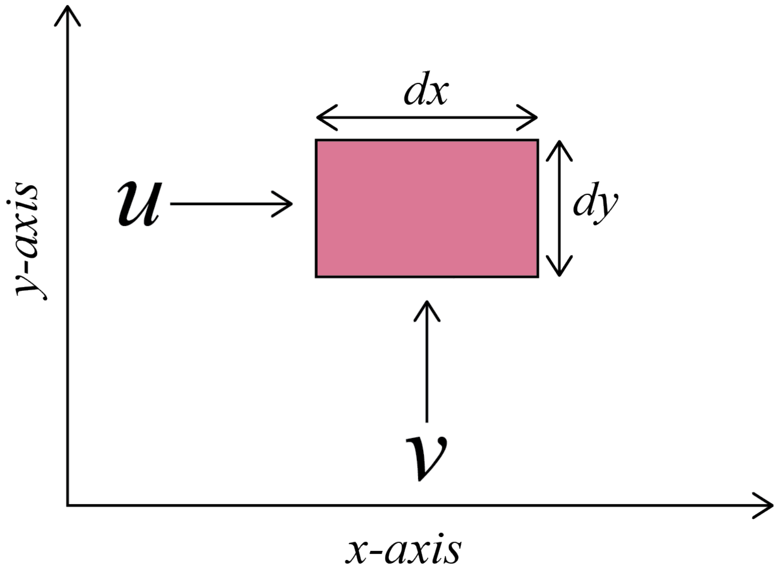
Stream function is a mathematical device which describes the form of any particular pattern of flow.

It is shows by Greek letter, ψ , called “sai”.

Stream function is volume rate of flow across any line connecting point A and B.

$$Q = AV \quad \Longrightarrow \quad V = \frac{Q}{A}$$

For Cartesian coordinate:

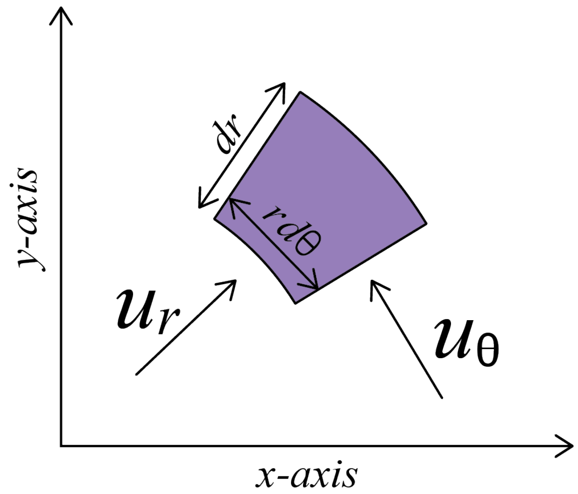


$$u = \frac{d\psi}{dy}$$

$$v = -\frac{d\psi}{dx}$$

Negative sign need to be added, because the rotation direction is counter-clockwise.

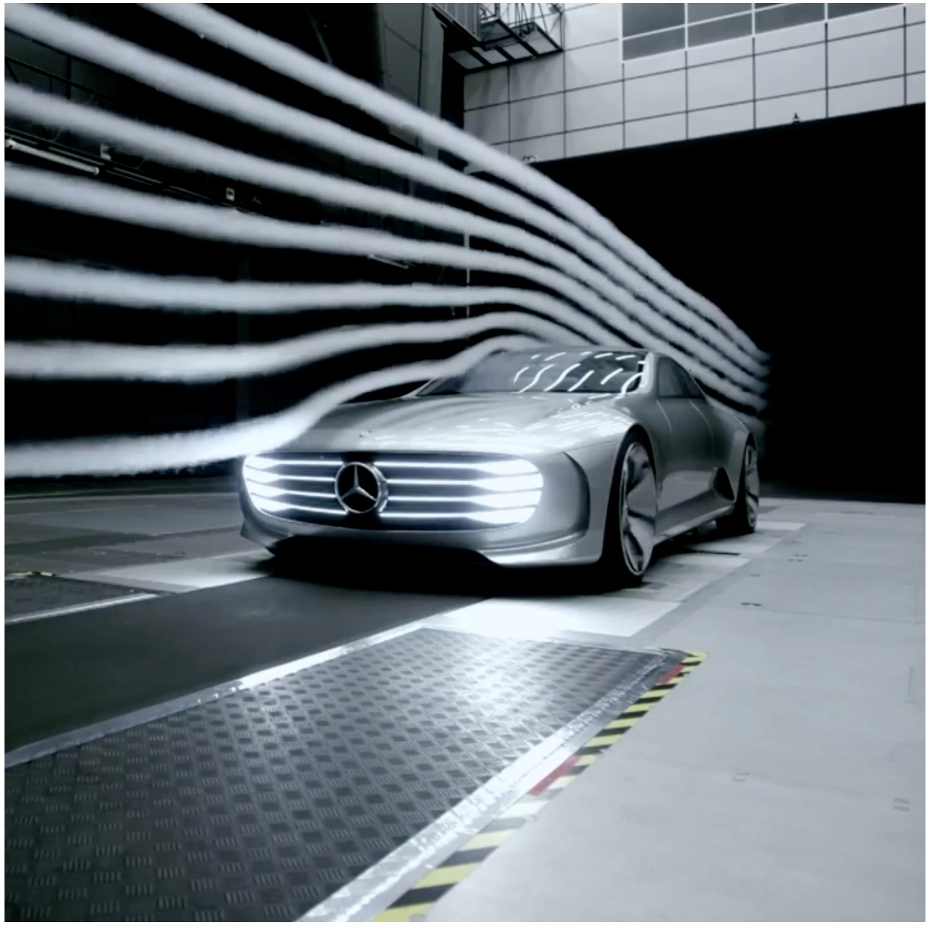
For polar coordinate:

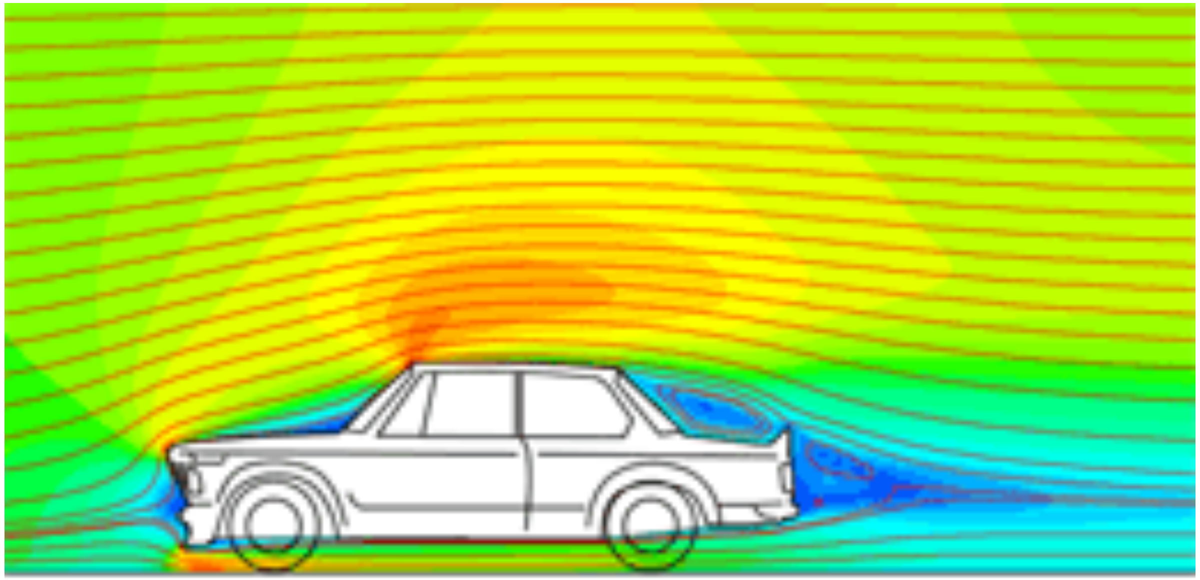


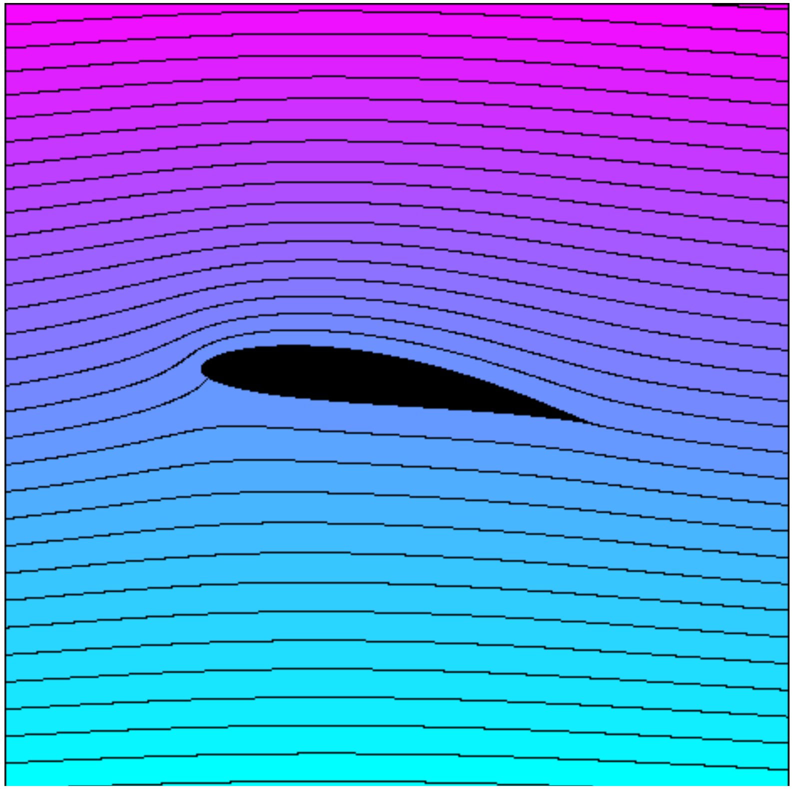
$$u_r = \frac{d\psi}{r d\theta} \qquad u_\theta = -\frac{d\psi}{dr}$$

Negative sign need to be added, because the rotation direction is counter-clockwise.

Streamline or ψ (sai) can be proved in experimental works.







https://www.youtube.com/watch?v=-H0UuVdbjls

https://www.youtube.com/watch?v=eLCya5p3LGs

IDEAL FLOW

VELOCITY POTENTIAL

CIRCULATION AND VORTICITY

CIRCULATION, Γ

Circulation, for a closed contour in a fluid is defined as the line integral evaluated along the contour of the component of the velocity that is locally tangent to the contour.

It is designed as capital gamma, Γ .

$$\Gamma = \oint u \cdot dL$$

$u =$ component velocity
$L =$ distance travelled by the fluid

Circulation can be considered as the amount of force that push along a closed boundary or path.
In the other words, circulation is the total "push" you get when going along a path, such as a circle.

VORTICITY, ζ

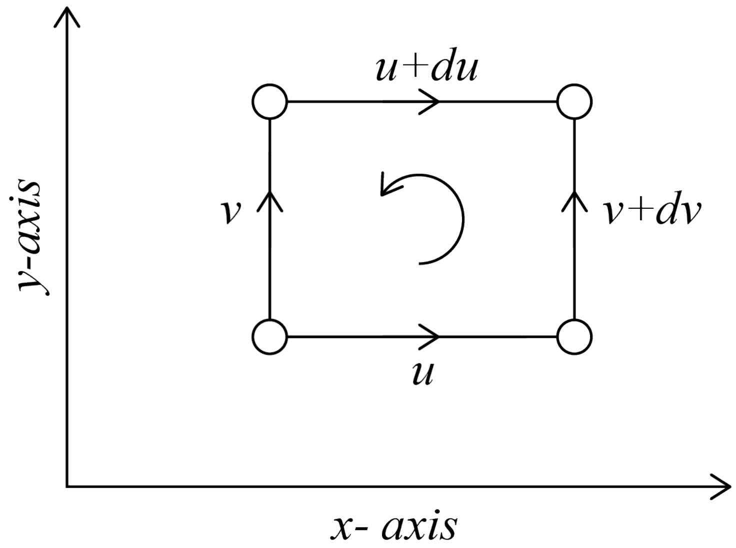
Vorticity is shown by Greek symbol – zeta, ζ

It is defined as circulation per unit area.

$$\text{Vorticity}, \zeta = Z = \frac{\text{Circulation}, \Gamma}{\text{Area}}$$

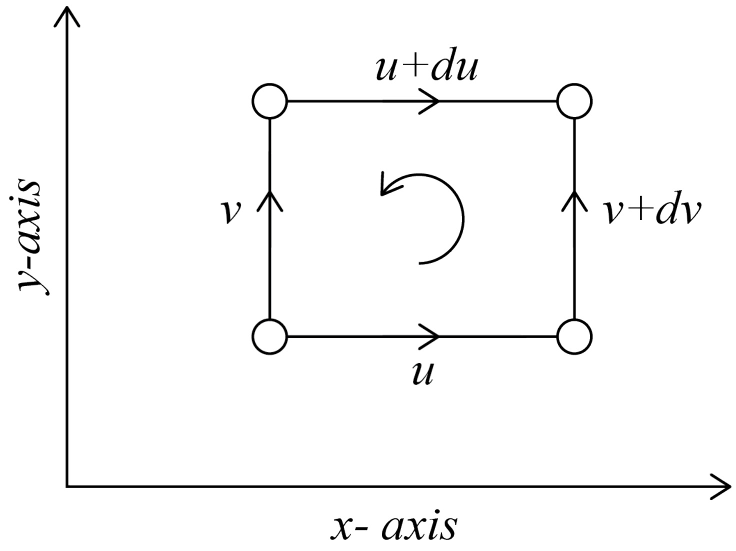
Vorticity is the tendency for elements of the fluid to “spin”.

CIRCULATION AND VORTICITY IN CARTESIAN COORDINATE



$$\Gamma = (u)dx + (v + dv)dy - (u + du)dx - (v)dy$$

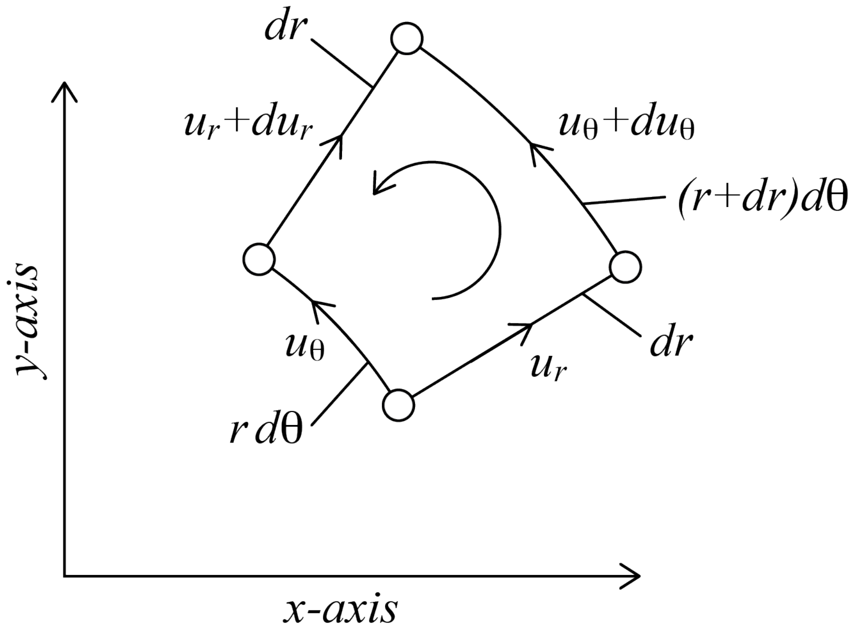
$$\Gamma = (dv)dy - (du)dx$$



$$\zeta = \frac{\Gamma}{A} = \frac{(dv)dy - (du)dx}{dx \cdot dy}$$

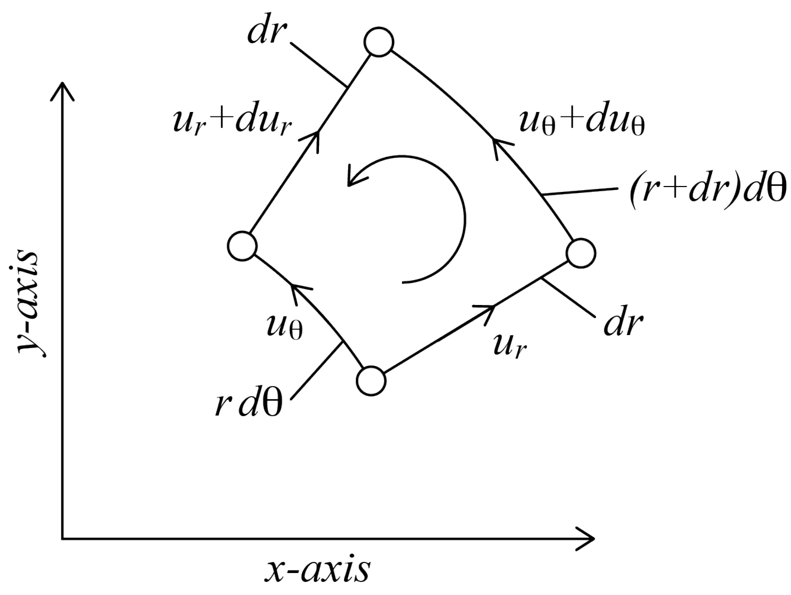
$$\zeta = \frac{dv}{dx} - \frac{du}{dy}$$

CIRCULATION AND VORTICITY IN POLAR COORDINATE



$$\Gamma = (u_r)dr + (u_\theta + du_\theta)(r + dr)d\theta - (u_r + du_r)dr - (u_\theta)rd\theta$$

$$\Gamma = (u_\theta)drd\theta + (du_\theta)rd\theta - du_r dr$$



$$\zeta = \frac{\Gamma}{A} = \frac{(u_\theta)drd\theta + (du_\theta)rd\theta - du_r dr}{dr(rd\theta)}$$

$$\zeta = \frac{u_\theta}{r} + \frac{du_\theta}{dr} - \frac{du_r}{rd\theta}$$

Not like continuity equation, vorticity is not always equal to zero.

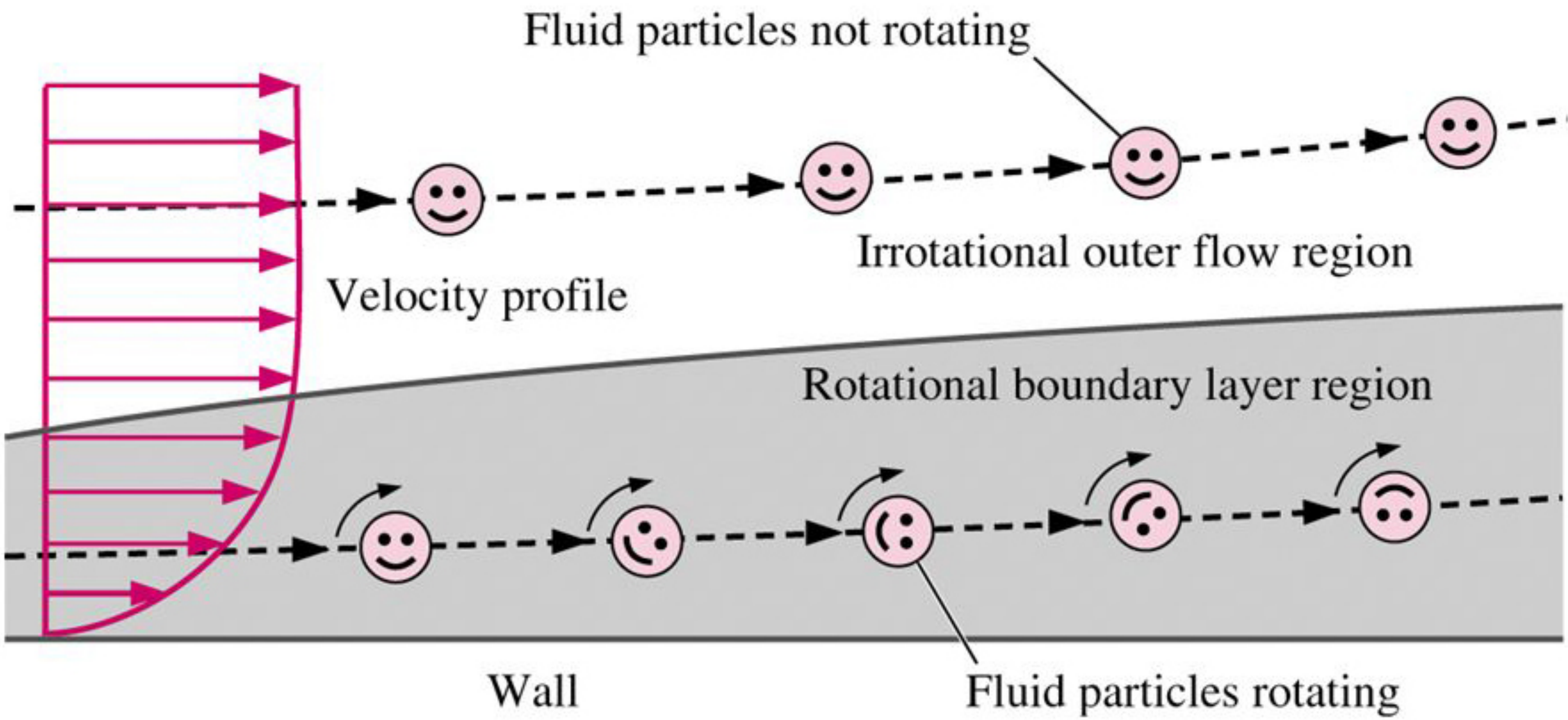
$$\frac{du}{dx} + \frac{dv}{dy} = 0$$

If ,

$$\zeta = 0 \quad \Rightarrow \quad \text{Irrotational flow}$$

If ,

$$\zeta \neq 0 \quad \Rightarrow \quad \text{Rotational flow}$$


Fluid particles not rotating
Irrotational outer flow region
Velocity profile
Rotational boundary layer region
Wall
Fluid particles rotating

VELOCITY POTENTIAL, ϕ

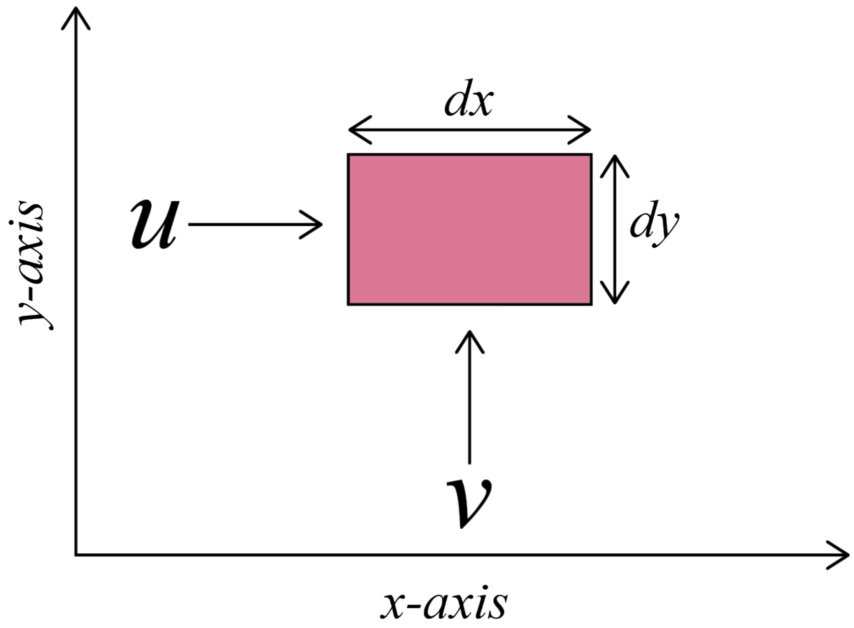
A velocity potential is a scalar potential used in potential flow theory. It was introduced by Joseph-Louis Lagrange in 1788.

It is denoted as phi, ϕ .

$$\phi = (\text{Velocity}) \times (\text{Distance travelled by the fluid})$$

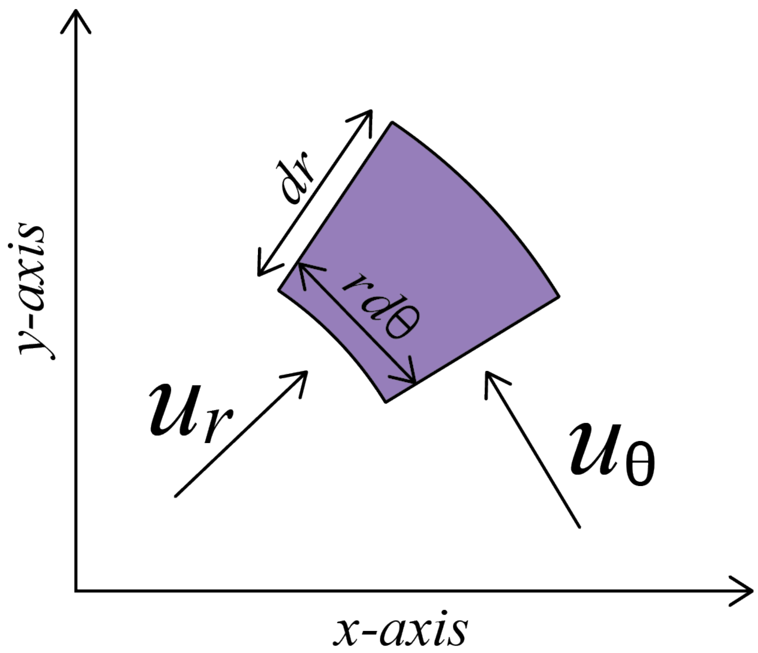
$$(\text{Velocity}) = \frac{\phi}{(\text{Distance travelled by the fluid})}$$

For Cartesian coordinate:

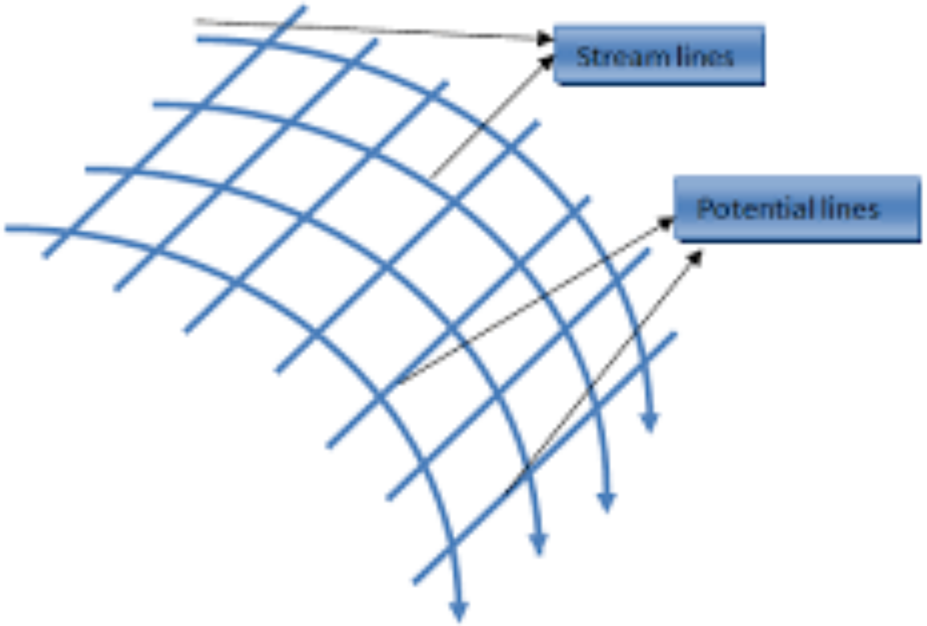


$$u = \frac{d\phi}{dx} \qquad v = \frac{d\phi}{dy}$$

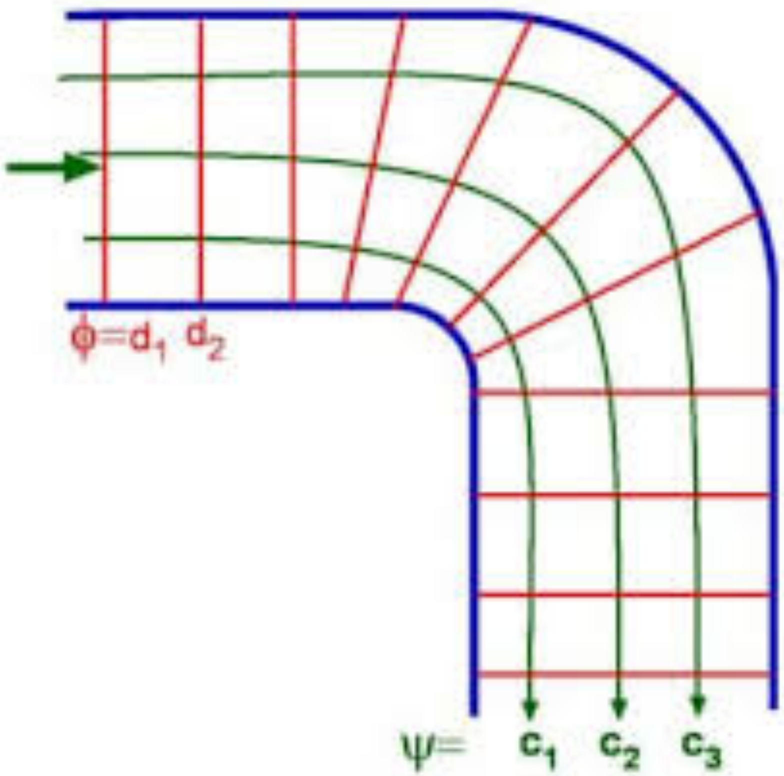
For polar coordinate:

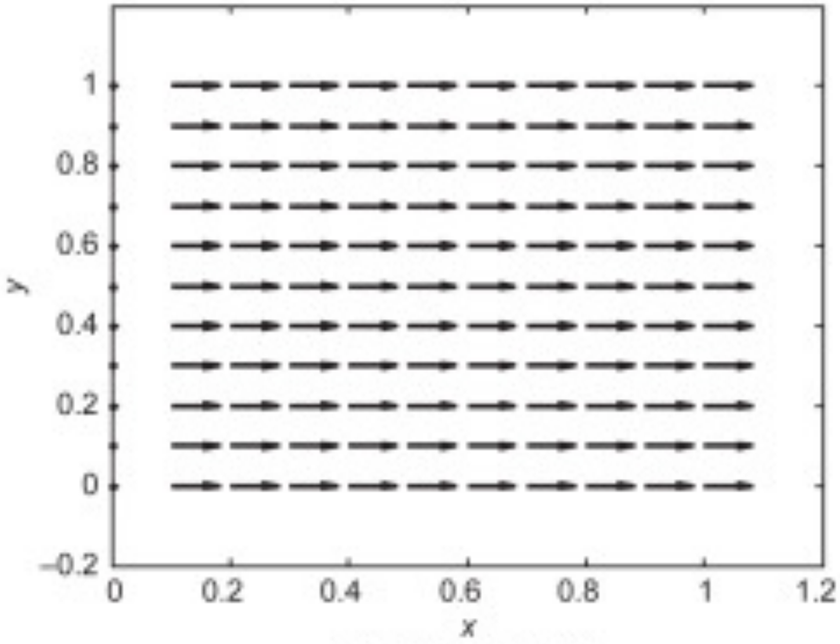


$$u_r = \frac{d\phi}{dr} \qquad u_\theta = \frac{d\phi}{r d\theta}$$

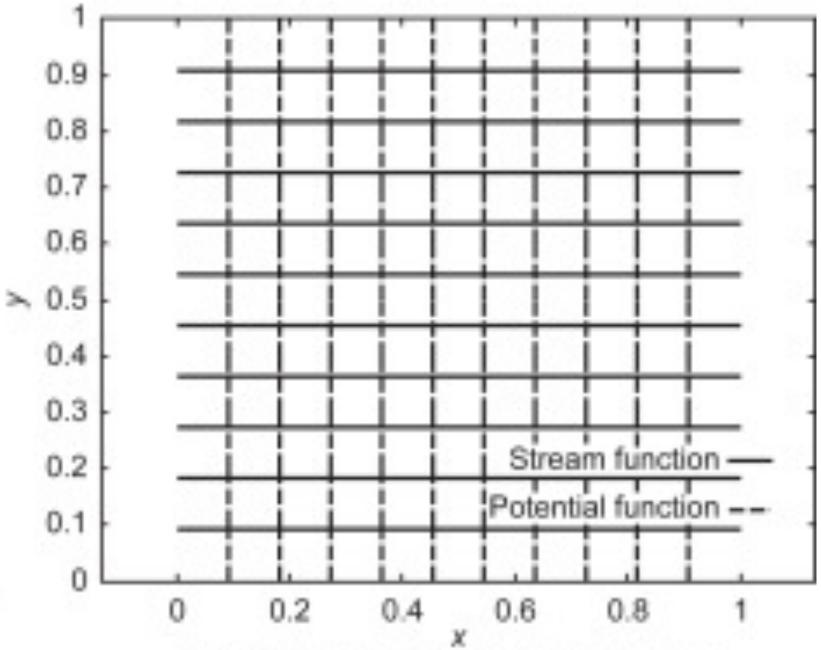


Flow Net

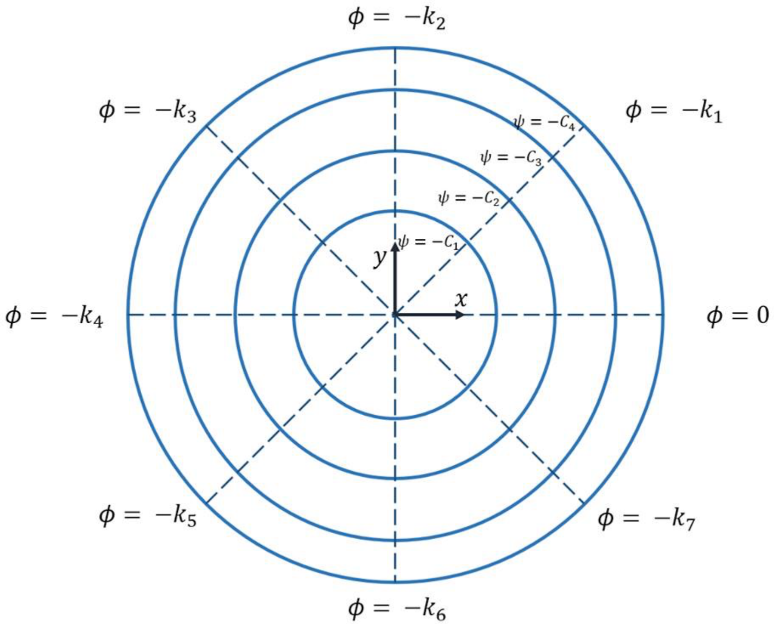

φ=d1 d2
ψ= c1 c2 c3

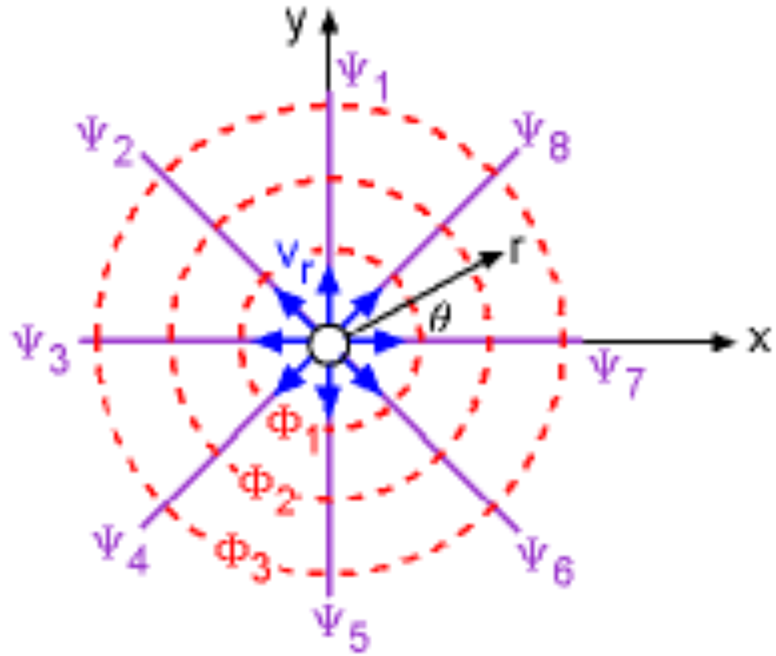


(Velocity distribution)



(Stream and potential function contours)


$\phi = -k_2$
$\phi = -k_3$
$\phi = -k_1$
$\psi = -C_4$
$\psi = -C_3$
$\psi = -C_2$
$\psi = -C_1$
y
x
$\phi = -k_4$
$\phi = 0$
$\phi = -k_5$
$\phi = -k_7$
$\phi = -k_6$


y
Ψ_1
Ψ_2
Ψ_8
v_r
r
θ
Ψ_3
x
Ψ_7
Φ_1
Φ_2
Φ_3
Ψ_4
Ψ_6
Ψ_5

CAUCHY-RIEMANN EQUATIONS

The Cauchy–Riemann equations, named after Augustin Cauchy and Bernhard Riemann.

Cartesian coordinate:	
$u = \frac{d\psi}{dy} = \frac{d\phi}{dx}$	$v = -\frac{d\psi}{dx} = \frac{d\phi}{dy}$

Polar coordinate:	
$u_r = \frac{d\psi}{rd\theta} = \frac{d\phi}{dr}$	$u_\theta = -\frac{d\psi}{dr} = \frac{d\phi}{rd\theta}$

TUTORIAL

Question 1

Prove whether the flow field below satisfies the continuity equation.

(a) $u = 2x \qquad v = -2y$

(b) $u = 2xy + y^2 t \qquad v = xy + x^2 t$

Answer 1

Knows that:

$$\frac{du}{dx} + \frac{dv}{dy} = 0$$ Continuity equation

a)

$$u = 2x, \qquad v = -2y$$

$$\frac{du}{dx} = 2 \qquad \frac{dv}{dy} = -2$$

$$\frac{du}{dx} + \frac{dv}{dy} = 2 - 2 = 0$$ Satisfies the continuity equation

b)

$$u = \ 2xy + y^2t, \qquad\qquad v = \ xy + x^2t$$

$$\frac{du}{dx} = \ 2y \qquad\qquad \frac{dv}{dy} = \ x$$

$$\frac{du}{dx} + \frac{dv}{dy} = 2y + x \neq 0$$

Not satisfies the continuity equation

Question 2

Stream function is given as:

$$\psi = x^2 + 2x + 4y^2$$

Determine the component velocity of u and v

Answer 2

$$u = \frac{d\psi}{dy} = 8y$$

$$v = -\frac{d\psi}{dx} = -(2x + 2)$$

Question 3

Stream function is given as, $\psi = 3x^2y - y^3$

Determine whether the flow field is rotational or irrotational. If it is irrotational flow, calculate its velocity component and resultant velocity.

Answer 3

We can use this equation to check whether it is rotational or irrotational flow.

$$\zeta = \frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = 0$$

$\psi = 3x^2y - y^3$

$$\frac{d\psi}{dx} = 6xy \qquad \frac{d\psi}{dy} = 3x^2 - 3y^2$$

$$\frac{d^2\psi}{dx^2} = 6y \qquad \frac{d^2\psi}{dy^2} = -6y$$

$\zeta = 6y + (-6y) = 0$

Flow is irrotational

$$\psi = 3x^2y - y^3$$

$$u = \frac{d\psi}{dy} = 3x^2 - 3y^2 \qquad v = -\frac{d\psi}{dx} = -6xy$$

Halaju paduan $\quad q = \sqrt{u^2 + v^2}$

$$= [(3x^2 - 3y^2)^2 + (-6xy)^2]^{\frac{1}{2}}$$

$$= 3(x^2 + y^2)$$

$$= 3(r^2)$$

$$(r^2 = x^2 + y^2)$$

You may also use this equation to check it:

$$\zeta = \frac{dv}{dx} - \frac{du}{dy}$$

Question 4

One flow field has component velocity as shown below. Determine the stream function and velocity potential.

$$u = x - 4y, \quad v = -y - 4x$$

Answer 4

To determine ψ

$$u = \frac{d\psi}{dy}$$

$$d\psi = udy$$

Substitute the value of u and integrate both side

$$\psi = \int (x - 4y)dy$$

$$\psi = xy - 2y^2 + f(x) + c$$

Condition $x = 0, \; y = 0 \rightarrow \; c = 0$, we know that:

$$\psi = xy - 2y^2 + f(x) \qquad \text{(Eq.1)}$$

$$\psi = xy - 2y^2 + f(x) \qquad \text{(Eq.1)}$$

Differentiate both side with dx

$$\frac{d\psi}{dx} = y + \frac{df(x)}{dx} = -v = -(-y - 4x)$$

$$y + \frac{df(x)}{dx} = y + 4x$$

$$\frac{df(x)}{dx} = 4x$$

$$\int df(x) = \int 4x \cdot dx$$

$$f(x) = 2x^2$$

Substitute into (Eq.1)

$$\psi = xy - 2y^2 + 2x^2$$

To determineϕ

$$u = \frac{d\phi}{dx}$$

$$d\phi = udx$$

Integrate both side

$$\int d\phi = \int udx$$

$$\phi = \int (x - 4y)dx$$

$$= \frac{x^2}{2} - 4xy + f(y) + c$$

Condition $x = 0,\ y = 0 \rightarrow\ c = 0$

$$\phi = \frac{x^2}{2} - 4xy + f(y)$$

To determine $f(y)$

$$\frac{d\phi}{dy} = -4x + \frac{df(y)}{dy} = v = -y - 4x$$

$$-4x + \frac{df(y)}{dy} = -y - 4x$$

$$\frac{df(y)}{dy} = -y$$

$$f(y) = \frac{-y^2}{2}$$

$$\phi = \frac{x^2}{2} - 4xy - \frac{y^2}{2}$$

Stream function (ψ) and velocity potential (ϕ) must be perpendicular to each other.

It can be proved that the gradient yield of $\left(\frac{dy}{dx}\right)$ for the stream function and the velocity potential yield a value of -1 (negative one)

$$\left(\frac{dy}{dx}\right)_{\psi} = \frac{dy}{d\psi} \times \frac{d\psi}{dx} = \frac{1}{u} \times (-v) = -\frac{v}{u}$$

$$\left(\frac{dy}{dx}\right)_{\phi} = \frac{dy}{d\phi} \times \frac{d\phi}{dx} = \frac{1}{v} \times (u) = \frac{u}{v}$$

Thus;

$$\left(\frac{dy}{dx}\right)_{\psi} \times \left(\frac{dy}{dx}\right)_{\phi} = \left(-\frac{v}{u}\right)\left(\frac{u}{v}\right) = -1$$

Tamat