CHAPTER 2:
FUNDAMENTAL EQUATIONS OF COMPRESSIBLE VISCOUS FLOW

Dimensionless parameters in viscous flow

Since our basic equations of motion are extremely difficult to analyze in general, we should cast them in their most efficient form, thereby increasing the usefulness of whatever solutions we find.

This accomplished by non-dimensionalizing the equations and boundary conditions.

For simplicity, assume constant c_p and c_v , approximately true for all gases, and also neglect the second coefficient of viscosity λ , which is seldom needed. Then our four variables, p, ρ, V and T , will depend upon space and time and eight parameters that occur in the basic equations and boundary conditions:

$$(V) \text{ or } (p) \text{ or } (\rho) \text{ or } (T) = f\left(x_i, t, \mu, k, g, c_p, T_w, q_w, \ell, \mathcal{T}\right)$$

Above mention parameters are assumed to be known from data or thermodynamics state relations. Gravity is constant. Now, we need to select constant reference properties appropriate to the flow:

1. Reference velocity U (the free stream velocity)
2. Reference length L (body length for external flow or duct diameter for internal flow)
3. Free stream properties $p_0, \rho_0, T_0, \mu_0, k_0,$

Steady viscous flows have no characteristic time of their own, so select particle residence time $\frac{L}{U}$ as a reference time. If the flow oscillates with frequency ω , one might select ω^{-1}as a reference time.

Now define dimensionless variables and denote them by asterisk.

$x^* = \frac{x}{L}$	$t^* = \frac{tU}{L}$	$p^* = \frac{p}{\rho_o U^2}$
$\Phi^* = \frac{L^2}{\mu_o U^2}\Phi$	$\rho^* = \frac{\rho}{\rho_o}$	$T^* = \frac{T - T_0}{T_w - T_0}$
$\mu^* = \frac{\mu}{\mu_o}$	$k^* = \frac{k}{k_o}$	$\nabla^* = L\nabla$
$V^* = \frac{V}{U}$		

Dimensionless for continuity equation:

Mass will always constant

$$m = \rho \forall = \text{constant}$$

$$\frac{Dm}{Dt} = \frac{D}{Dt}(\rho \forall) = 0 = \rho \frac{D\forall}{Dt} + \forall \frac{D\rho}{Dt} \tag{1}$$

Relation between normal-strain rate can be written is:

$$\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = \frac{1}{\forall}\frac{D\forall}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \text{div}\, V = \nabla V$$

$$\forall\ \text{div}\, V = \frac{D\forall}{Dt} \tag{2}$$

Substitute (2) into (1):

$$0 = \rho\forall \operatorname{div} V + \forall \frac{D\rho}{Dt}$$

$$0 = \frac{D\rho}{Dt} + \rho \operatorname{div} V$$

(or)

$$0 = \frac{\partial\rho}{\partial t} + \operatorname{div} \rho V \qquad [2]$$

If incompressible flow, $\operatorname{div} V = 0$
which equivalent to requiring particles of constant volume

$$Q = \text{constant} = A_1V_1 = A_2V_2$$

Thus, continuity equation can be written as:

$$0 \;= \frac{\partial \rho}{\partial t} + \mathrm{div}\, \rho V = \frac{\partial \rho}{\partial t} + \nabla \rho V \tag{3}$$

Define the dimensionless variables:

$\rho^* = \dfrac{\rho}{\rho_o}$	$t^* = \dfrac{tU}{L}$	$\nabla^* = L\nabla$	$V^* = \dfrac{V}{U}$
$\rho = \rho^* \rho_o$	$t = \dfrac{t^* L}{U}$	$\nabla = \dfrac{\nabla^*}{L}$	$V = V^* U$

Substitute dimensionless variables into (1):

$$0 = \frac{\partial(\rho^* \rho_o)}{\partial\left(\frac{t^* L}{U}\right)} + \frac{\nabla^*}{L}(\rho^* \rho_o)(V^* U)$$

$$0 = \frac{\rho_o U}{L}\left(\frac{\partial \rho^*}{\partial t^*}\right) + \frac{\rho_o U}{L}\nabla^*(\rho^* V^*)$$

$$0 = \frac{\partial \rho^*}{\partial t^*} + \nabla^*(\rho^* V^*) \qquad (2)$$

From (2), there are no dimensionless parameter could be produced.

Dimensionless for Navier-Stokes equation:

$$\rho \frac{dV}{dt} = \rho g - \nabla p + \mu \nabla^2 V \quad (1)$$

Define the dimensionless parameter:

$\rho^* = \frac{\rho}{\rho_o}$	$V^* = \frac{V}{U}$	$t^* = \frac{tU}{L}$	$\mu^* = \frac{\mu}{\mu_o}$
$\nabla^* = L\nabla$	$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} z$		$p^* = \frac{p}{\rho_o U^2}$

Substitute in (1) :

$$(\rho^* \rho_o) \frac{d(V^* U)}{d\left(\frac{t^* L}{U}\right)} = (\rho^* \rho_o) g - \frac{\nabla^*}{L} (p^* \rho_o U^2) + (\mu^* \mu_o) \left(\frac{\nabla^*}{L}\right)^2 (V^* U)$$

$$\frac{\rho_o U^2}{L} \left(\rho^* \frac{dV^*}{dt^*}\right) = \rho_o g \rho^* - \frac{\rho_o U^2}{L} (\nabla^* p^*) + \frac{\mu_o U}{L^2} \left(\mu^* \nabla^{*2} V^*\right)$$

$$\left(\text{divide all with } \frac{\rho_o U^2}{L}\right)$$

$$\rho^* \frac{dV^*}{dt^*} = \frac{\rho_o g L}{\rho_o U^2}\rho^* - \frac{\rho_o U^2}{L}\frac{L}{\rho_o U^2}(\nabla^* p^*) + \frac{\mu_o U}{L^2}\frac{L}{\rho_o U^2}\left(\mu^* \nabla^{*2} V^*\right)$$

$$\rho^* \frac{dV^*}{dt^*} = \frac{gL}{U^2}\rho^* - \nabla^* p^* + \frac{\mu}{\rho_o U L}\left(\mu^* \nabla^{*2} V^*\right)$$

$$\rho^* \frac{dV^*}{dt^*} = \frac{1}{F_r}\rho^* - \nabla^* p^* + \frac{1}{Re}\left(\mu^* \nabla^{*2} V^*\right)$$

$$\left[F_r = \frac{U^2}{gL}\right] \; ; \; \left[Re = \frac{\rho_o U L}{\mu}\right]$$

Solution for 2D Navier-Stokes paksi- x

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Senaraikan semua dimensionless parameter (reference value):

$\rho^* = \frac{\rho}{\rho_o}$	$u^* = \frac{u}{u_o}$	$v^* = \frac{v}{u_o}$	$x^* = \frac{x}{L}$
$p^* = \frac{p}{\rho_o u^2}$	$\rho^* = \frac{\rho}{\rho_o}$	$\mu^* = \frac{\mu}{\mu_o}$	$y^* = \frac{y}{L}$

$$u = u^* u_o$$

$$\frac{\partial u}{\partial x} = \frac{\partial (u^* u_o)}{\partial (x^* L)} = \frac{u_o}{L} \frac{\partial u^*}{\partial x^*}$$

$$\frac{\partial u}{\partial y} = \frac{u_o}{L} \frac{\partial u^*}{\partial y^*}$$

$$\frac{\partial p}{\partial x} = \frac{\partial (p^* \rho_o {u_o}^2)}{\partial (x^* L)} = \frac{\rho_o {u_o}^2}{L} \frac{\partial p^*}{\partial x^*}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{u_o}{L}\frac{\partial u^*}{\partial x^*}\right)$$

$$= \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial x}\left(\frac{u_o}{L}\frac{\partial u^*}{\partial x^*}\right)$$

$$= \frac{\partial}{\partial x^*}\frac{\partial u^*}{\partial x^*}\frac{\partial x^*}{\partial x}\frac{u_o}{L}$$

$$= \frac{1}{L}\frac{u_o}{L}\frac{\partial^2 u^*}{\partial x^*} = \frac{u_o}{L^2}\frac{\partial^2 u^*}{\partial {x^*}^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_o}{L^2}\frac{\partial^2 u^*}{\partial {y^*}^2}$$

$$x^* = \frac{x}{L}$$

$$\frac{\partial}{\partial x}(x^*) = \frac{\partial}{\partial x}\left(\frac{x}{L}\right) = \frac{1}{L}$$

$$\frac{\partial x^*}{\partial x} = \frac{1}{L}$$

$$\frac{\partial y^*}{\partial x} = \frac{1}{L}$$

LHS:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = p^*\rho_o\left(u^*u_o\frac{u_o}{L}\frac{\partial u^*}{\partial x^*}+v^*u_o\frac{u_o}{L}\frac{\partial u^*}{\partial y^*}\right)$$

$$= \frac{\rho_o{u_o}^2}{L}p^*\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)$$

RHS:

$$-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = -\frac{\rho_o{u_o}^2}{L}\frac{\partial p^*}{\partial x^*}+\mu^*\mu_o\left(\frac{u_o}{L^2}\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{u_o}{L^2}\frac{\partial^2 u^*}{\partial {y^*}^2}\right)$$

$$= -\frac{\rho_o{u_o}^2}{L}\frac{\partial p^*}{\partial x^*}+\mu^*\mu_o\frac{u_o}{L^2}\left(\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{\partial^2 u^*}{\partial {y^*}^2}\right)$$

LHS = RHS

$$\frac{\rho_o{u_o}^2}{L}p^*\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{\rho_o{u_o}^2}{L}\frac{\partial p^*}{\partial x^*}+\mu^*\mu_o\left(\frac{u_o}{L^2}\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{u_o}{L^2}\frac{\partial^2 u^*}{\partial {y^*}^2}\right)$$

$$\left(\text{Divide through by } \frac{\rho_o{u_o}^2}{L}\right)$$

$$p^*\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial x^*}+\frac{\mu_o}{\rho_o u_o L}\mu^*\left(\frac{\partial^2 u^*}{\partial {x^*}^2}+\frac{\partial^2 u^*}{\partial {y^*}^2}\right) \qquad \left[\frac{1}{Re}=\frac{\mu_o}{\rho_o u_o L}\right]$$