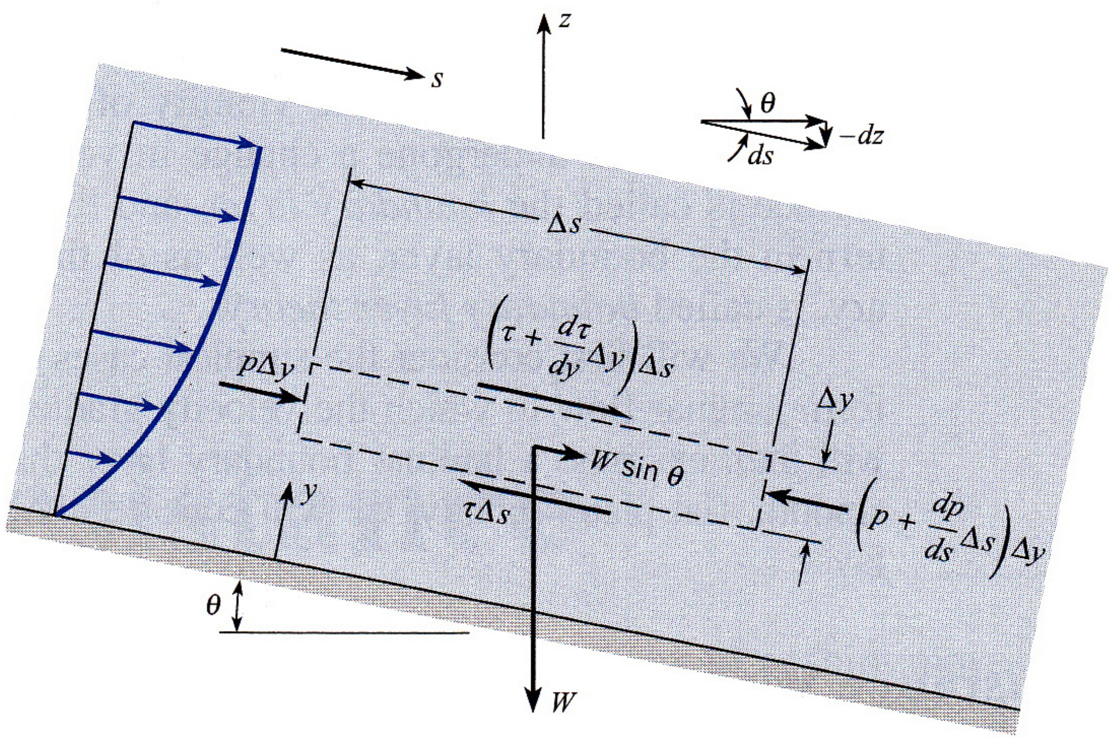
CHAPTER 3
SOLUTIONS OF THE NEWTONIAN VISCOUS FLOW EQUATIONS

SURFACE RESISTANCE WITH UNIFORM LAMINAR FLOW (COUETTE FLOW)

In this section, we consider three two-dimensional laminar flows with parallel stream lines. The flows are steady and uniform; that is, there is no change in velocity along a stream line. Using the momentum equation, we will first derive a general equation for the flow velocity and then apply it to three specific problem.

1. Flow produced by a moving plate
2. Liquid flow down an inclined plane
3. Flow between two stationary parallel plate



Consider the control volume as shown above, which aligned with the flow in direction s. The streamlines are inclined at an angle θ with respect to the horizontal plane. The control volume has dimension $\Delta s \times \Delta y \times unity$; that is, the control volume has a unit length into the screen.

By application of the momentum equation, the sum of the forces acting in the s-direction is equal to the net outflow of momentum from the control volume.

The flow is uniform, so that outflow of momentum is equal to the in-flow and the momentum equation reduces to

$$\sum F_s = 0$$

There are three forces acting on the matter in the control volume: the forces due to pressure, shear stress and gravity.

The net pressure force is:

$$p\Delta y - \left(p + \frac{dp}{ds}\Delta s\right)\Delta y = -\frac{dp}{ds}\Delta s\Delta y$$

The net force due to shear stress is:

$$\left(\tau + \frac{d\tau}{dy}\Delta y\right)\Delta s - (\tau\Delta s) = \frac{d\tau}{dy}\Delta y\Delta s$$

The component of gravitational force is $\rho g \Delta s \Delta y \sin\theta$. However, $\sin\theta$ can be related to the rate at which the elevation, z , decreases with increasing s and is given by $-\frac{dz}{ds}$.

Thus, the gravitational force becomes:

$$\rho g \Delta s \Delta y \sin\theta = -\rho g \Delta s \Delta y \left(\frac{dz}{ds}\right)$$

Summing all forces to zero and dividing through by $\Delta s \Delta y$ results in:

$$\frac{d\tau}{dy} = \frac{d}{ds}(p + \rho g z) = \frac{d}{ds}(p + \gamma z)$$

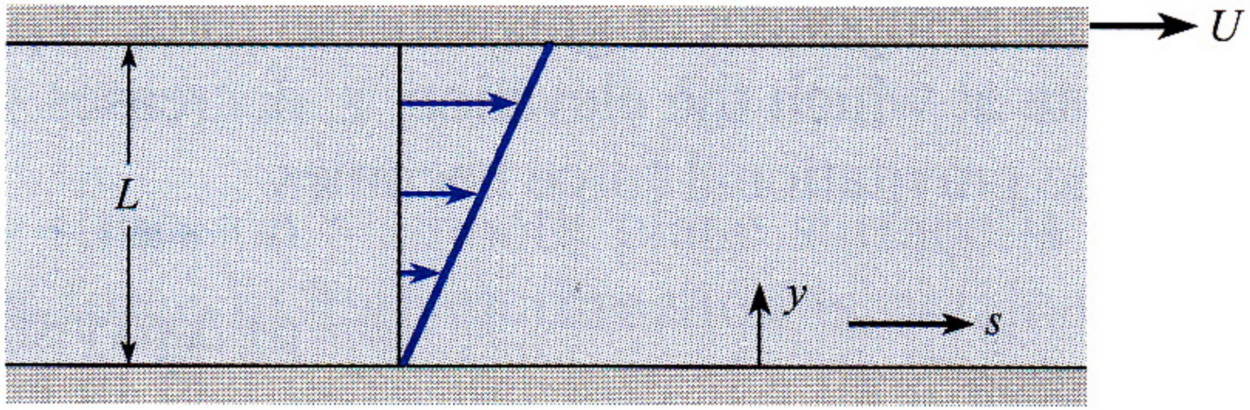
where we note that the gradient of the shear stress is equal to the change in piezometric pressure in the flow direction. The shear stress is equal to $\mu\left(\frac{du}{dy}\right)$, so the equation becomes:

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{d}{ds}(p + \rho g z) = \frac{1}{\mu}\frac{d}{ds}(p + \gamma z) \qquad (1)$$

We will apply this equation to three flow configurations.

FLOW PRODUCED BY MOVING PLATE

Consider the flow between the two plates as shown below. The lower plate is fixed and the upper plate is moving with a speed U. The plates are separated by a distance L. In this situation, there is no pressure gradient in the flow direction $\left(\frac{dp}{ds} = 0\right)$ and the stream lines are in the horizontal direction $\left(\frac{dz}{ds} = 0\right)$



Eq.(1) can be written as:

$$\frac{d^2u}{dy^2} = 0$$

The two boundary conditions are:

$$u = 0 \quad \text{at} \quad y = 0$$

$$u = U \quad \text{at} \quad y = L$$

Integrating this equation twice gives:

$$u = C_1 y + C_2$$

Applying the boundary conditions gives:

$$u = U\left(\frac{y}{L}\right)$$

which shows that the velocity profile is linear between two plates. The shear stress is constant and equal to:

$$\tau = \mu\left(\frac{du}{dy}\right) = \mu\left(\frac{U}{L}\right)$$

This flow is known as a Couette flow after a French scientist, M. Couette, who did pioneering work on the flow between parallel plates and rotating cylinders.

Example:

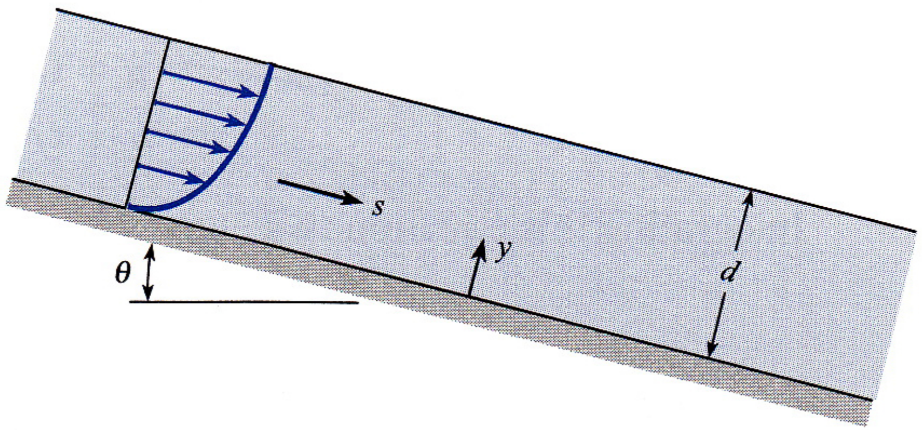
If the fluid between the plates is SAE 30 lubricating oil at $T = 38°C$, the plates are spaced 0.3mm apart, the upper plate is moved at a velocity of 1.0m/s, what is the surface resistance for 1.0m^2 of the upper plate?

$$\tau = \mu\left(\frac{du}{dy}\right) = \mu\left(\frac{U}{L}\right) = (0.1)\frac{1}{0.0003} = 333\ (N/m^2)$$

$$F = \tau A = 333\ \ N$$

LIQUID FLOW DOWN AN INCLINED PLANE

Consider the flow down an inclined plane as shown below. The liquid layer, which has a depth d , has a free surface where the pressure is constant, so along the surface $\left(\frac{dp}{ds} = 0\right)$.



The pressure is hydrostatic across any section in the z-direction but does not change in the stream line direction. The shear stress at the free surface between the air and liquid is small and therefore neglected.

In this section, the gravitational force is balanced by the shear stress. The differential equation, Eq.(1), reduces to:

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{d}{ds}(p + \gamma z) = -\frac{\gamma}{\mu}\sin\theta$$

The boundary conditions are:

$$u = 0 \quad \text{at} \quad y = 0$$

$$\frac{du}{dy} = 0 \quad \text{at} \quad y = d$$

Integrating this equation once gives:

$$\frac{du}{dy} = (-y)\left(\frac{\gamma}{\mu}\right)\sin\theta + C$$

Applying boundary conditions at $y = d$ shows:

$$C = \left(\frac{\gamma d}{\mu}\right)\sin\theta$$

The equation becomes:

$$\frac{du}{dy} = \left(\frac{\gamma}{\mu}\right) \sin\theta\,(d - y)$$

Integrating the second time results in:

$$u = \left(\frac{\gamma}{\mu}\right) \sin\theta \left(yd - \frac{y^2}{2}\right) + C$$

and the constant of integration is set equal to zero to satisfy the boundary condition at $y = 0$

This equation can be written as:

$$u = \left(\frac{\gamma \sin\theta}{2\mu}\right)(2yd - y^2) = \left(\frac{g \sin\theta}{2\upsilon}\right)(2yd - y^2)$$

$\upsilon =$ kinematic viscosity
$\mu =$ dynamic viscosity

The resulting profile is a parabola with maximum velocity occurring at the free surface. The maximum velocity is:

$$u_{max} = \left(\frac{\gamma d^2}{2\mu}\right) \sin\theta$$

The discharge per unit width can be obtained by integrating the velocity u over the depth of flow:

$$q = \int_0^d u \cdot dy = \left(\frac{\gamma \sin\theta}{2\mu}\right)\left[dy^2 - \frac{y^3}{3}\right]_0^d = \frac{1}{3}\left(\frac{\gamma \sin\theta}{\mu}\right) d^3$$

The average velocity, $V,$ can be obtained by:

$$V = \frac{q}{d} = \frac{1}{3}\left(\frac{\gamma \sin\theta}{\mu}\right) d^2 = \frac{g d^2}{3\upsilon} \sin\theta$$

$$d = \text{cross sectional area}$$
$$q = \text{discharge}$$

The slope, $S_0 = \tan\theta$, is approximately equal to $\sin\theta$ for a small slopes. Sometimes, equation for velocity can be written as:

$$V = \frac{gd^2 S_0}{3\upsilon}$$

Experiments have shown that if Reynolds number based on the depth of the flow,

$$Re = \frac{\rho V d}{\mu} = \frac{V d}{\upsilon}$$

is less than 500, one can expect laminar flow in this situation. If the Reynolds number is greater than 500, the flow become turbulent and the results of this section are no longer valid.

Example:

Crude oil, $\upsilon = 9.3\times10^{-5}$ m2/s, $SG = 0.92$, flows over a flat plate that has a slope of $S_0 = 0.02$. If the depth of flow is 6mm, what is the maximum velocity and what is the discharge per meter of width of plate? Determine the Reynolds number for this flow.

$$u = \left(\frac{gS_0}{2\upsilon}\right)(2yd - y^2)$$

At $y = d$; $u_{max} = 0.038 \ \ m/s$

Discharge per meter of width is:

$$q = \frac{1}{3}\left(\frac{\gamma \sin\theta}{\mu}\right)d^3 = \frac{gS_0 d^3}{3\upsilon} = 1.52\times10^{-4} \ \ m^2/s$$

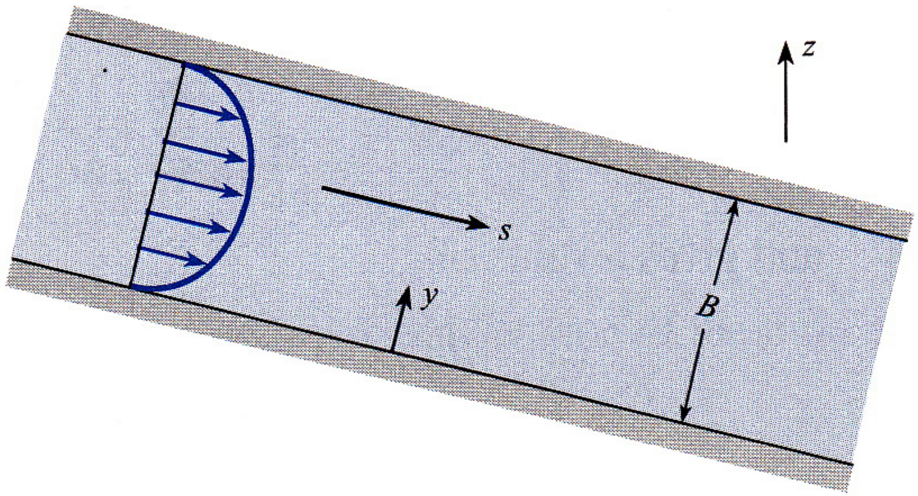
Reynolds number:

$$Re = \frac{Vd}{\upsilon} = 1.63$$

(The flow is laminar)

FLOW BETWEEN STATIONARY PARALLEL PLATES

Consider the two parallel plates separated by a distance B as shown below.



In this situation, the flow velocity is zero at the surface for both plates. The boundary conditions become:

$$u = 0 \quad \text{at} \quad y = 0$$

$$u = 0 \quad \text{at} \quad y = B$$

The gradient in piezometric pressure is constant along the stream line. Integrating Eq.(1) twice gives:

$$u = \frac{y^2}{2\mu}\frac{d}{ds}(p + \gamma z) + C_1 y + C_2$$

To satisfy the boundary condition at $y = 0$, we need to set $C_2 = 0$. Applying the boundary condition at $y = B$ requires that C_1 is:

$$C_1 = -\frac{B}{2\mu}\frac{d}{ds}(p + \gamma z)$$

The final equation for the velocity is:

$$u = -\frac{1}{2\mu}\frac{d}{ds}(p + \gamma z)(By - y^2) = -\frac{\gamma}{2\mu}(By - y^2)\frac{dh}{ds}$$

which is parabolic profile with the maximum velocity occurring on the centreline between the plates $\left(y = \frac{B}{2}\right)$.

The maximum velocity is:

$$u_{max} = -\left(\frac{B^2}{8\mu}\right)\frac{d}{ds}(p + \gamma z)$$

or in terms piezometric head:

$$u_{max} = -\left(\frac{B^2\gamma}{8\mu}\right)\frac{dh}{ds}$$

The fluid always flows in the direction of decreasing piezometric pressure or piezometric head, so $\frac{dh}{ds}$ is negative.

This giving a positive value for u_{max} .

The discharge per unit width can be find by integrating the velocity over the distance between plates:

$$q = \int_0^B u \cdot dy = -\left(\frac{B^3}{12\mu}\right)\frac{d}{ds}(p + \gamma z) = -\left(\frac{B^3}{12\mu}\right)\frac{dh}{ds}$$

The average velocity is:

$$V = \frac{q}{B} = -\left(\frac{B^2}{12\mu}\right)\frac{d}{ds}(p + \gamma z) = \frac{2}{3}u_{max}$$

As was the case with unconfined laminar liquid flow over a plane surface, the velocity distribution is parabolic. However, in this situation, the maximum velocity occurs midway between two plates. Note that flow is result of a change of the piezometric head, not just change of p or z alone.

Experiments reveal that if Reynolds number is less than 1000, the flow is laminar. For Reynolds number greater than 1000, the flow may be turbulent and the equations in this section invalid.

Example:

Oil having a specific gravity of 0.8 and a viscosity of 0.02 N.s/m2 flows downward between two vertical smooth plates spaced 10mm apart. If the discharge per meter of width is 0.01 m2/s, what is the pressure gradient $\frac{dp}{ds}$ for this flow.

Reynolds number:

$$Re = \frac{VB}{\upsilon} = \frac{\rho VB}{\mu} = \frac{\rho q}{\mu} = 400$$

(Flow is laminar)

$$\upsilon = \frac{\mu}{\rho} = 0.000025 \quad m^2/s$$

The gradient in piezometric head can be written as:

$$\frac{dh}{ds} = -\frac{12\mu q}{B^3\gamma} = -\frac{12\upsilon q}{B^3 g} = -0.306$$

However,

$$\frac{dh}{ds} = \frac{d}{ds}\left(\frac{p}{\gamma} + z\right) = -0.306$$

Because the plates are vertically oriented and s is positive downward, $\frac{dz}{ds} = -1$,

Thus,

$$\frac{d\left(\frac{p}{\gamma}\right)}{ds} = 1 - 0.306$$

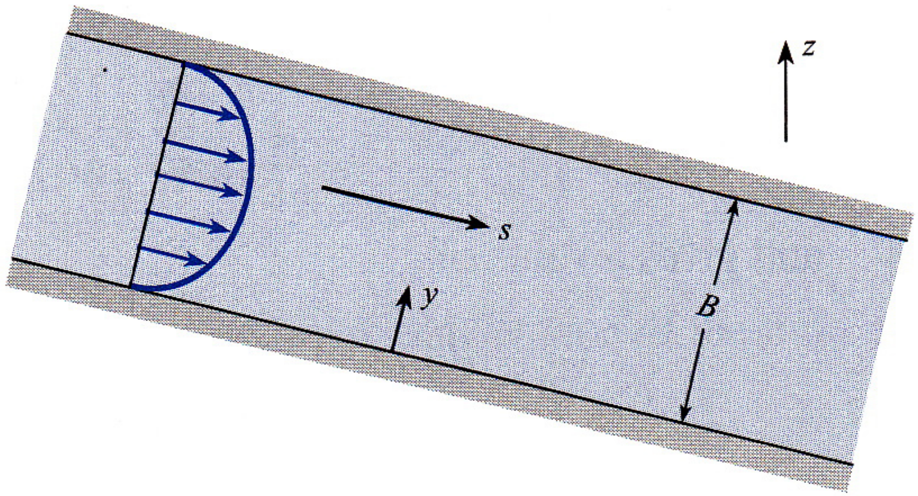
or

$$\frac{dp}{ds} = 5447 \quad N/m^2$$

In the other word, the pressure is increasing downward at a rate of 5.45 kPa per meter of length of plate.

FULLY DEVELOPED FLOW BETWEEN PARALLEL PLATES USING NAVIER-STOKES EQUATIONS

The flow field between parallel plates will be derived here using the continuity and Navier-Stokes equations. Reference is made to above mention figure, where y is the coordinate normal to the plates and x is the flow direction (same as s-direction).



The flow field is fully developed (uniform), so the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ are zero.

Also the flow is steady, so $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$ are zero.

The component of the gravity vector in the x-direction is $g \sin\theta$ and
The component of the gravity vector in the y-direction is $-g \cos\theta$.

The continuity equation for planar incompressible flow is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Since $\frac{\partial u}{\partial x} = 0$, the continuity equation reduces to:

$$\frac{\partial v}{\partial y} = 0$$

or

$$v = \text{constant}$$

At the surface of the plate $(y = 0)$, the velocity is zero. So,

$$v = 0$$

everywhere in the flow field.

The Navier-Stokes equation in the y-direction is:

$$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \rho g \cos\theta$$

Because v is zero everywhere, there is no acceleration of the fluid in the y-direction, so this component of the Navier-Stokes equation reduces to:

$$\frac{\partial p}{\partial y} = -\rho g \cos\theta$$

Integrating over y , one has

$$p = -y\rho g \cos\theta + p_{y=0}(x)$$

where $p_{y=0}(x)$ is the pressure distribution along the lower wall. This equation shows that the pressure decreases with elevation in the duct, as expected. In fact, $y \cos\theta$ is equal to z , so this equation can be written as the equation for hydrostatic pressure variation, namely:

$$\rho + \rho g z = p_{y=0}(x)$$

The pressure gradient in the x-direction is:

$$\frac{\partial p}{\partial x} = \frac{\partial p_{y=0}}{\partial x} = \frac{dp}{dx}$$

and is the same for all values of y across the duct for any value of x.

The Navier-Stokes equation in the x-direction is:

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho g \sin\theta$$

For steady, fully developed flow the left-hand-side of this equation reduces to zero (no acceleration in the x-direction), and the equation becomes:

$$\frac{\partial p}{\partial x} - \rho g \sin\theta = \mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

Because u is a function of y only $\left(\frac{\partial u}{\partial x} = 0\right)$ and $\frac{\partial p}{\partial x}$ is a function only of x , this equation becomes:=

$$\frac{dp}{dx} - \rho g \sin\theta \; = \; \mu\left(\frac{d^2u}{dy^2}\right)$$

The slope of the duct can be expressed as:

$$\sin\theta \; = \; -\frac{dz}{dx}$$

Previously, we know that :

$$\mu\left(\frac{d^2u}{dy^2}\right) = \frac{dp}{dx} - \rho g\left(-\frac{dz}{dx}\right) = \frac{d}{dx}(p + \rho g z)$$

or

$$\rho g\frac{dh}{ds} = \gamma\frac{dh}{ds} = \mu\left(\frac{d^2u}{dy^2}\right)$$

which is the same equation as used in the previous section to obtain the velocity distribution.