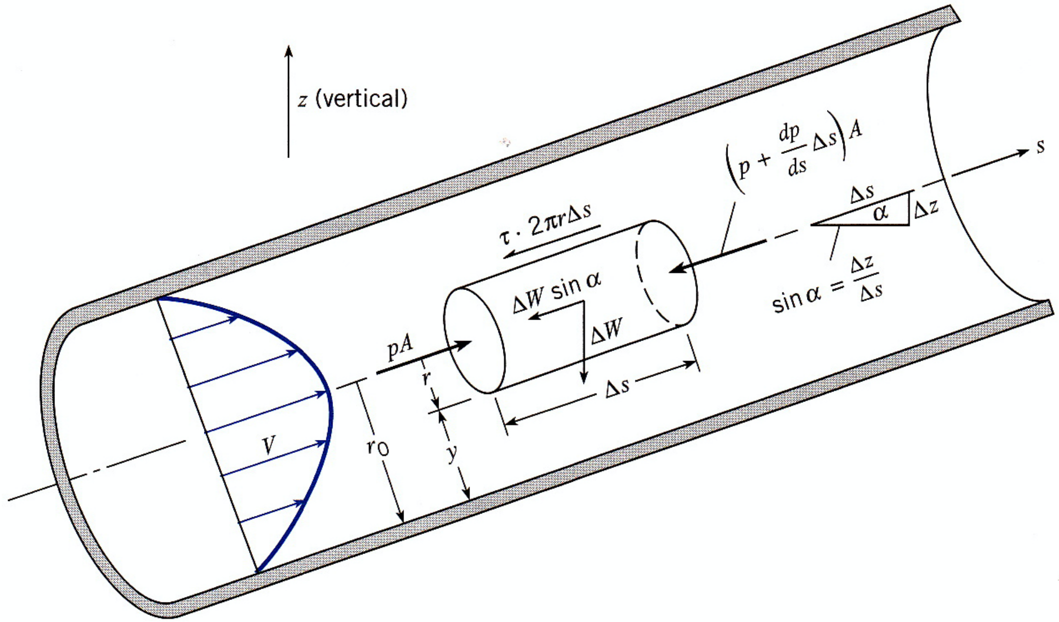
CHAPTER 3
SOLUTIONS OF THE NEWTONIAN VISCOUS FLOW EQUATIONS

FLOW IN CONDUITS
SHEAR STRESS DISTRIBUTION ACROSS A PIPE SECTION

The velocity distribution in a pipe is directly linked to the shear stress distribution; hence it is important to understand the latter. To determine the shear stress distribution, we start with the equation of equilibrium applied to a cylindrical control volume that is oriented coaxially with the pipe, as shown below.



For the condition above, it is assumed that the flow is uniform (stream lines are straight and parallel). Therefore, the net momentum flow through the control volume is zero.

Also the pressure across any section of the pipe will be hydrostatically distributed. Thus, the pressure force acting on an end face of the fluid element will be the product of the pressure at the center of the element (also at the center of the pipe) and the area of the face of the element.

With steady uniform flow, equilibrium between the pressure, gravity and shearing forces acting on the fluid will prevail. Consequently, the momentum equation yields the following:

$$\sum F_s = 0$$

$$pA - \left(p + \frac{dp}{ds}\Delta s\right)A - \Delta W \sin\alpha - \tau(2\pi r)\Delta s = 0 \tag{1}$$

Here;

$$\Delta W = \gamma A \Delta s \qquad \sin\alpha = \frac{dz}{ds}$$

Eq.(1) becomes:

$$-\frac{dp}{ds}\Delta sA - \gamma A\Delta s\frac{dz}{ds} - \tau(2\pi r)\Delta s = 0 \tag{2}$$

Then, when we divide Eq.(2) through by ΔsA and simplify, we could obtain,

$$\tau = \frac{r}{2}\left[-\frac{d}{ds}(p + \gamma z)\right] \tag{3}$$

Since the gradient itself, $\frac{d}{ds}(p + \gamma z)$ is negative and constant across the section for uniform flow, it follows that $\frac{d}{ds}(p + \gamma z)$ will be positive and constant across the pipe section.

Thus, τ in Eq.(3) will be zero at the center of the pipe and will increase linearly to a maximum at the pipe wall.

We will use Eq.(3) in the following section to derive the velocity distribution for laminar flow.

Laminar flow in pipes

We determine how the velocity varies across the pipe by substituting for τ in Eq.(3) its equivalent $\mu \frac{dV}{dy}$ and integrating.

First, making the substitution, we have,

$$\mu \frac{dV}{dy} = \frac{r}{2}\left[-\frac{d}{ds}(p+\gamma z)\right] \tag{4}$$

Because $\frac{dV}{dy} = -\frac{dV}{dr}$, Eq.(4) becomes,

$$\frac{dV}{dr} = -\frac{r}{2\mu}\left[-\frac{d}{ds}(p+\gamma z)\right] \tag{5}$$

When we separate variables and integrate across the section, we obtain,

$$V = -\frac{r^2}{4\mu}\left[-\frac{d}{ds}(p+\gamma z)\right] + C \tag{6}$$

We can evaluate the constant of integration in Eq.(6) by noting that when $r = r_0$, the velocity $V = 0$. Therefore, the constant of integration is given by:

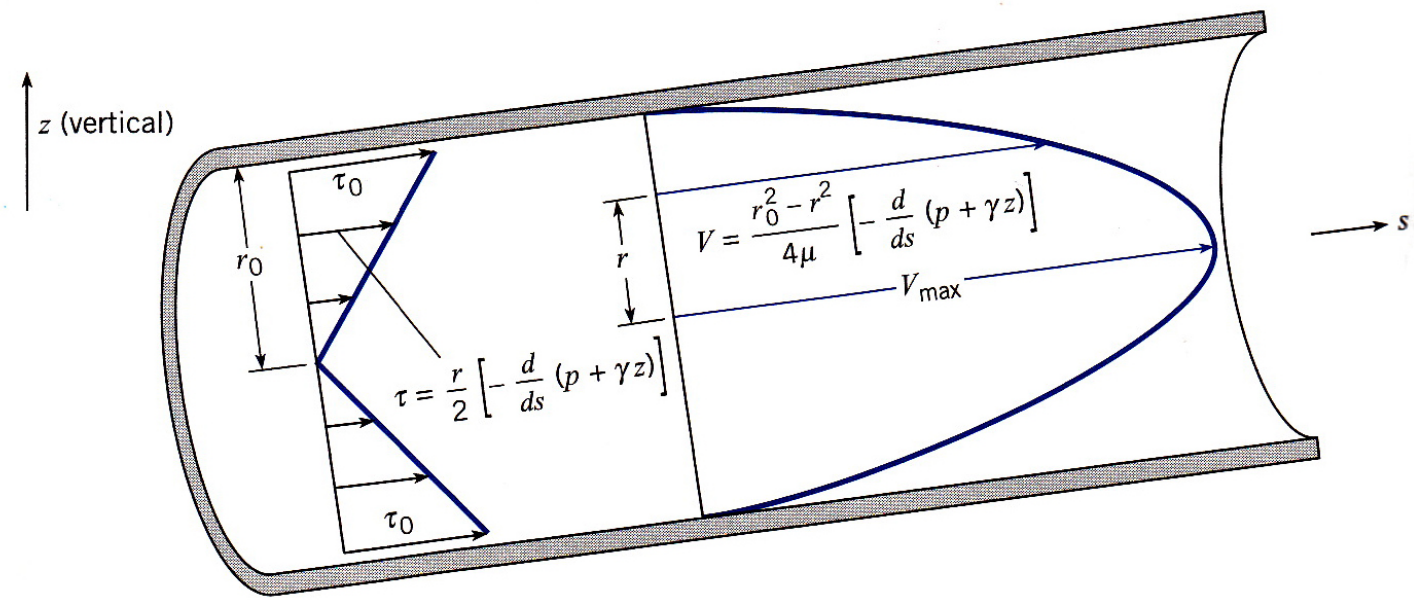
$$C = \left(\frac{r_0^2}{4\mu}\right)\left[-\frac{d}{ds}(p + \gamma z)\right]$$

and Eq.(6) becomes;

$$V = \frac{r_0^2 - r^2}{4\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] \tag{7}$$

Eq.(7) indicates that the velocity distribution for laminar flow in pipe is parabolic across the section with the maximum velocity at the center of the pipe. Figure below shows the variation of the shear stress and velocity in the pipe.

Laminar flow in a round pipe is known as Hagen-Pouseuille flow, named after a German, Hagen and a Frenchman, Poiseuille, who studied low-speed in tubes in the 1840s.



For many problems, we wish to relate the pressure change to the rate of flow or mean velocity $\bar{V}$, in the conduit. Therefore, it is necessary to integrate $dQ = V \cdot dA$ over the cross sectional area of flow. That is:

$$Q = \int V \cdot dA = \int_0^{r_0} \frac{r_0^2 - r^2}{4\mu} \left[-\frac{d}{ds}(p + \gamma z) \right] (2\pi r dr)$$

The factor $\frac{\pi}{4\mu}\left[\frac{d(p+\gamma z)}{ds}\right]$ is constant across the pipe section. Therefore, upon integration, we could obtain:

$$Q = \frac{\pi}{4\mu}\left[\frac{d}{ds}(p + \gamma z)\right]\left[\frac{(r^2 - r_0^2)}{2}\right]_0^{r_0}$$

which reduces to

$$Q = \frac{\pi r_0^4}{8\mu}\left[-\frac{d}{ds}(p + \gamma z)\right]$$

If we divided through by the cross sectional area of the pipe, we have an expression for the mean velocity:

$$\bar{V} = \frac{r_0^2}{8\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] \tag{1}$$

From here, we could conclude as:

$$\bar{V} = \frac{V_{max}}{2}$$

Substitute $r_0 = \frac{D}{2}$

$$\bar{V} = \frac{D^2}{32\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] \quad \text{or}$$

$$\frac{d}{ds}(p + \gamma z) = -\frac{32\mu\bar{V}}{D^2} \tag{2}$$

Integrating Eq.(2) along the pipe between sections (1) and (2), we obtain;

$$p_2 - p_1 + \gamma(z_2 - z_1) = -\frac{32\mu\bar{V}}{D^2}(s_2 - s_1) \tag{3}$$

Here, $(s_2 - s_1)$ is the length L of pipe between the two sections. Therefore, Eq.(3) can be rewritten as:

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2 + \frac{32\mu L\bar{V}}{\gamma D^2} \qquad (4)$$

It can be seen that when the general energy equation for incompressible flow in conduits, is reduced to one for uniform flow in a constant-diameter pipe where $V_1 = V_2$, the results is:

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2 + h_f$$

Here, h_f is used instead of h_L to signify head loss due to frictional resistance of the pipe. It can be conclude that the head loss for laminar flow is given by:

$$h_f = \frac{32\mu L\bar{V}}{\gamma D^2} = \frac{32\mu LV}{\gamma D^2}$$

Here, the bar over the V has been omitted to conform to the standard practice of denoting the mean velocity in one-dimensional flow analyses by V without the bar.

Example:

Oil (with SG=0.90 and μ=0.5 N.s/m^2) flows steadily in a 3-cm pipe. The pipe is vertical and the pressure at an elevation of 100 m is 200 kPa. If the pressure at an elevation of 85 m is 250 kPa, is the flow direction up or down? What is the velocity at the center of the pipe and at 6 mm from the center, assuming that the flow is laminar?

First, determine the rate of change of $p+\gamma z$. Taking s in the z direction.

$$\frac{d}{ds}(p+\gamma z)=\frac{(p_{100}+\gamma z_{100})-(p_{85}+\gamma z_{85})}{(100-85)}=5.53 \quad kN/m^3$$

The quantity of $(p+\gamma z)$ is not constant with elevation. It increases upward (decreases downward). Therefore, the direction of flow is downward.

When $r = 0$ (at the center of pipe), the velocity will be maximum. Thus,

$$V_{center} = \frac{r_0^2 - r^2}{4\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] = \frac{r_0^2}{4\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] = -0.622 \quad m/s$$

The velocity at $r = 6\ mm = 0.006\ m$

$$V_{6mm} = \frac{r_0^2 - r^2}{4\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] = \frac{(0.015)^2 - (0.006)^2}{4\mu}\left[-\frac{d}{ds}(p + \gamma z)\right] = -0.522 \quad m/s$$

Example:

Oil with SG=0.85, with a kinematic viscosity of 6×10^{-4} m^2/s flows in a 15-cm pipe at a rate of 0.020 m^3/s. What is the head loss per 100-m length of pipe?

$$V = \frac{Q}{A} = 1.13 \quad m/s$$

$$Re = \frac{VD}{\upsilon} = 283 \qquad (Laminar)$$

Head loss for 100 m ($L = 100 \ m$)

$$h_f = \frac{32\mu LV}{\gamma D^2} = 9.83 \ m$$

The head loss is 9.83m per 100m of length.