CHAPTER 4
LAMINAR BOUNDARY LAYER

THERMAL BOUNDARY LAYER

Just as a velocity boundary layer develops when there is fluid flow over a surface, a thermal boundary layer must develop if the fluid free stream and surface temperatures differ. Consider flow over an isothermal flat plate, as shown below.

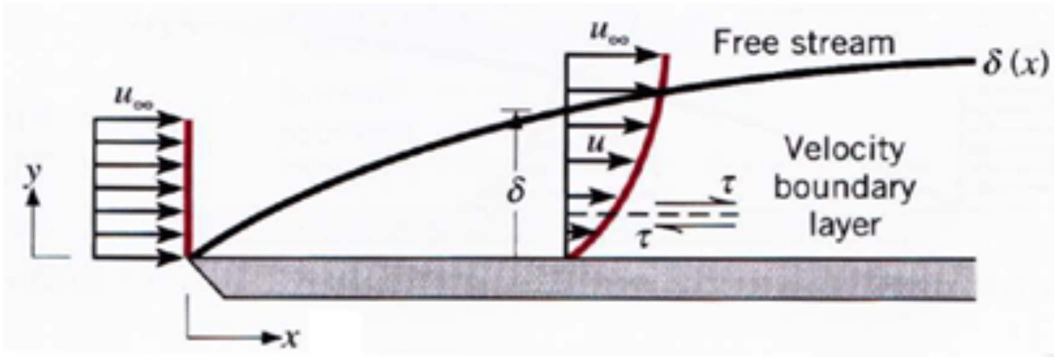
At the leading edge the temperature profile is uniform, with $T(y) = T_\infty$.

However, fluid particles that come into contact with the flat plate achieve thermal equilibrium at the plate's surface temperature. In turn, these particles exchange energy with those in the adjoining fluid layer, and temperature gradients develop in the fluid.

The region of the fluid in which these temperature gradients exist is the thermal boundary layer, and its thickness δ_t or δ_T is typically defined as the value of y for which the ratio

$$\frac{T_s - T}{T_s - T_\infty} = 0.99$$

With increasing distance from the leading edge, the effects of heat transfer penetrate further into the free stream and the thermal boundary layer grows.


u_∞
Free stream
$\delta(x)$
u_∞
y
u
Velocity
boundary
layer
δ
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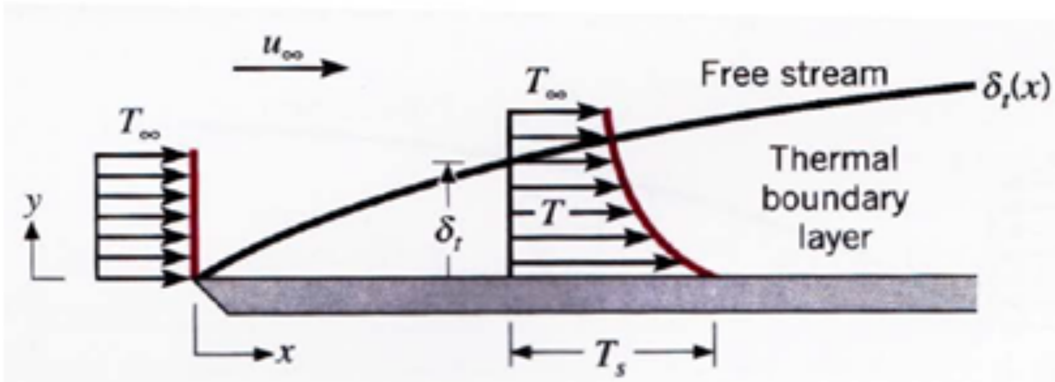

u_∞
T_∞
Free stream
$\delta_t(x)$
T_∞
Thermal
boundary
layer
y
δ_t
T
x
T_s

For laminar flow:

Thermal boundary layer thickness, δ_T :

$$\delta_T = \frac{\delta_V}{(Pr)^{\frac{1}{3}}} = \delta_V (Pr)^{-1/3}$$

$$Pr = \frac{\text{Viscous diffusion rate}}{\text{Thermal diffusion rate}} = \frac{\upsilon}{\alpha} = \frac{c_p \mu}{k}$$

where:

$$c_p = \text{Specific heat (J/kg.K)}$$
$$\mu = \text{Dynamic viscosity (Pa.s)}$$
$$k = \text{Thermal conductivity (W/m.K)}$$
$$\upsilon = \text{Kinematic viscosity } (\text{m}^2/s)$$
$$\alpha = \text{Thermal diffusivity } (\text{m}^2/s)$$

From Blasius velocity boundary layer thickness δ_V :

$$\delta_V = \frac{5x}{\sqrt{Re}}$$

Case 1:

If $Pr = 1$ $\delta_T = \delta_V$

The thermal boundary layer at any x is equals the thickness of the velocity boundary layer.

Case 2:

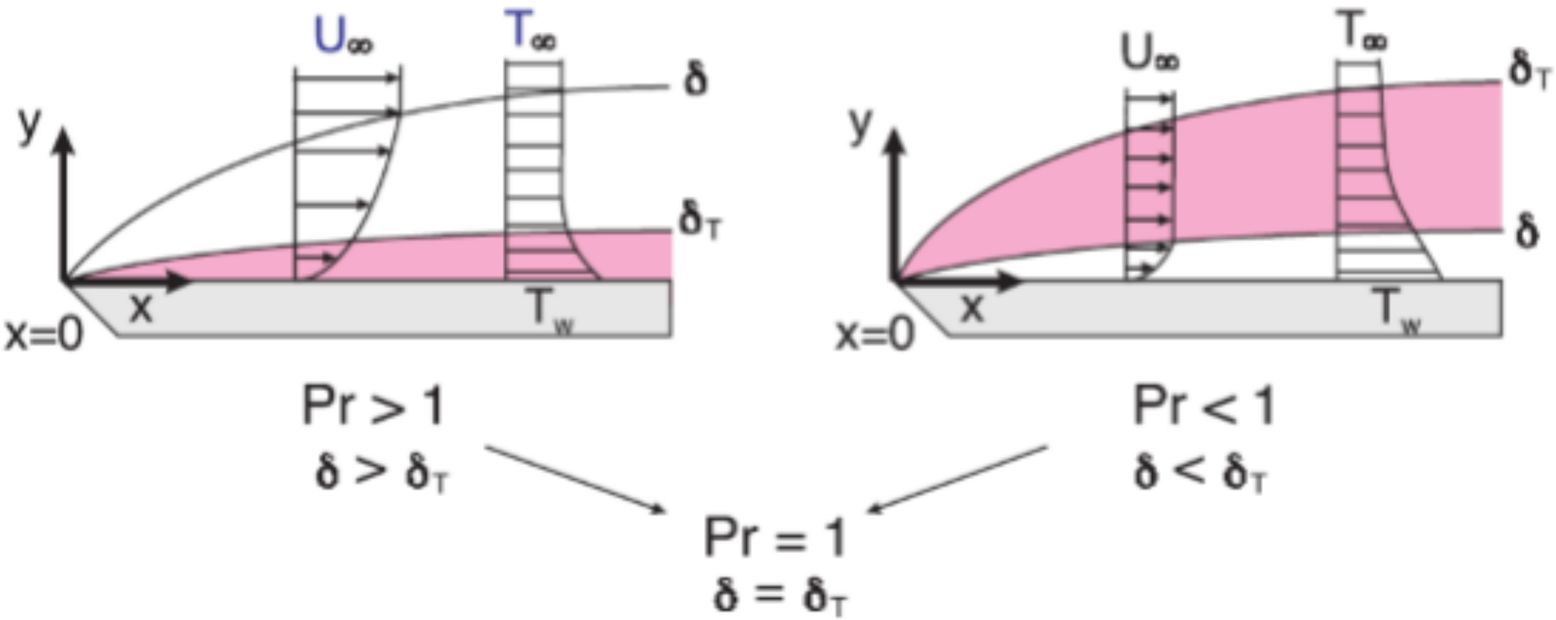
If $Pr < 1$ $\delta_T > \delta_V$

The thermal boundary layer at any x is thicker than the thickness of the velocity boundary layer.

Case 3:

If $Pr > 1$ $\delta_T < \delta_V$

The thermal boundary layer at any x is thinner than the thickness of the velocity boundary layer. For example, an oil with $Pr = 1000$ has $\delta_T \approx \frac{\delta_V}{10}$



Fluid	Kinematic viscosity [m2/s]	Prandl Number
Air	16.96 x 10^-6	0.699
Carbon dioxide	9.294 x 10^-6	0.76
Hydrogen	118.6 x 10^-6	0.684
Water	0.657 x 10^-6	4.34
Mercury	0.109 x 10^-6	0.0252
Glycerine	223 x 10^-6	2450

For turbulent flow:

The thermal boundary layer thickness for turbulent flow does not depend on the Prandtl number but instead on the Reynolds number.

$$\delta_T = \delta_V = \frac{0.37x}{\sqrt[5]{Re_x}} = 0.37x\left({Re_x}^{-1/5}\right)$$

This turbulent boundary layer thickness formula assumes:

(1) The flow is turbulent right from the start of the boundary layer.
(2) The turbulent boundary layer behaves in a geometrically similar manner.

Neither one of these assumptions is true for the general turbulent boundary layer case so care must be exercised in applying this formula.

Example 01:

Assuming that the thermal boundary layer of air is the same thickness as the momentum boundary layer. Calculate the thickness of the thermal boundary layer 1.00 cm away from the leading edge of a flat plate if the thermal diffusivity is $\alpha = 2.009\times10^{-5}\ m^2/s$ and the free stream velocity is $U = 7.0\ cm/s$.

$$Re = \frac{\rho VL}{\mu} = \frac{(1.23)(0.07)(0.01)}{1.8\times10^{-5}} = 47.83$$

The flow is laminar.

$$\delta_T = \frac{\delta_V}{(Pr)^{\frac{1}{3}}} = \delta_V (Pr)^{-1/3}$$

$$Pr = \frac{\upsilon}{\alpha} = \frac{1.47\times10^{-5}}{2.009\times10^{-5}} = 0.7317$$

$$\delta_V = \frac{5x}{\sqrt{Re}} = \frac{0.05}{6.916} = 0.00723$$

$$\delta_T = (0.00723)(0.7317)^{\frac{-1}{3}} = 0.00802 \quad (m)$$

Example 02:

Calculate the ratio of thermal boundary layer thickness to hydrodynamic boundary layer thickness for glycerine and hydrogen.

For glycerine: $Pr = 2450$

$$\frac{\delta_T}{\delta_V} = (2450)^{\frac{-1}{3}} = 0.07418$$

For hydrogen: $Pr = 0.684$

$$\frac{\delta_T}{\delta_V} = (0.684)^{\frac{-1}{3}} = 1.13496$$

Example 03:

For air at 15°C flowing over a flat plate at a free stream velocity of 6 m/s. Determine the velocity boundary layer and thermal boundary layer thickness at a distance of 0.5 m from the leading edge.

$$Re = \frac{\rho V L}{\mu} = \frac{(1.23)(6)(0.5)}{1.8\times10^{-5}} = 205{,}000$$

The flow is laminar.

$$\delta_T = \frac{\delta_V}{(Pr)^{\frac{1}{3}}} = \delta_V (Pr)^{-1/3}$$

$$Pr = 0.699$$

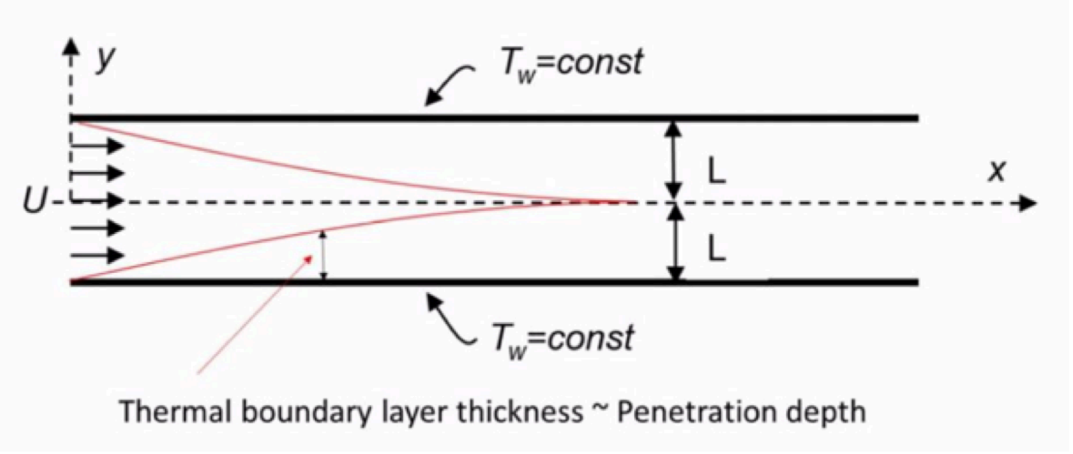
$$\delta_V = \frac{5x}{\sqrt{Re}} = \frac{2.5}{452.76926} = 0.0055216$$

$$\delta_T = (0.0055216)(0.699)^{\frac{-1}{3}} = 0.006221 \quad (m)$$

Example 04:

Find the thermal entry length for plug flow between two parallel plate with gap 2L (assumed that there is uniform heat flux).

Thermal entry length is defined as the location where the thermal boundary layer grown from the plate met at the centre line.



Thermal entry length for a flow between two plates at 2L is given by:
For uniform wall heat flux condition:

$$(L_T)_{\text{Laminar}} = 0.033(Re)(Pr)(D)$$

For uniform wall temperature condition:

$$(L_T)_{\text{Laminar}} = 0.043(Re)(Pr)(D)$$

$$D = \text{Distance between two plates.}$$

Entry length for a flow in a pipe:

$$(L_T)_{\text{Laminar}} = 0.05(Re)(Pr)(D)$$

$$(L_T)_{\text{Turbulent}} = 10(D)$$

Example 05:

Glycerin at 10°C is flowing over a flat plate at a free stream velocity of 2 m/s. Determine the velocity and thermal boundary layer thickness at a distance of 0.75 m from the leading edge. Also calculate the ratio of the velocity boundary layer thickness to the thermal boundary layer thickness for this flow and interpret the result.

Assumed that:

$$v = 0.0033421 \ m^2/s$$

$$Pr = 34000$$

$$Re = \frac{VL}{v} = 449$$

(Laminar)
Calculate the velocity boundary layer thickness:

$$\delta_V = \frac{5x}{\sqrt{Re}} = 0.177 \ m$$

Therefore, the velocity boundary layer thickness at a distance 0.75 m from the leading edge of the plate is 0.177 m.

Calculate the thermal boundary layer thickness:

$$\delta_T = \frac{\delta}{(Pr)^{\frac{1}{3}}} = 0.005464 \quad m$$

Therefore, the thermal boundary layer thickness at a distance 0.75 m from the leading edge of the plate is 0.005464 m.

Calculate the ratio of velocity boundary layer thickness to the thermal boundary layer thickness:

$$\frac{\delta_V}{\delta_T} = \frac{0.177}{0.005464} = 32.39$$

Conclusion: As the Prandtl number is very high for glycerine, the velocity boundary layer thickness is more than the thermal boundary layer thickness.

Example 06:

Engine oil at 100°C and a velocity of 0.1 m/s flows flows over both surface of a 0.8 m long flat plate maintained at 20°C. Determine:

a) Velocity boundary layer thickness at the trailing edge (m)
b) Thermal boundary layer thickness at the trailing edge (m)
c) The magnitude of the local heat flux at the trailing edge (W/m^2)
d) The local surface shear stress at the trailing edge (N/m^2)
e) The total drag force per unit width of the plate (N/m)
f) The magnitude of the heat transfer per unit width of the plate (W/m)

Properties of engine oil at $T_{oil} = \frac{T_s - T_\infty}{2} = 60°C$:

$$\rho = 864 \ \ kg/m^3$$
$$\upsilon = 8.61 \times 10^{-5} \, m^2/s$$
$$k = 0.14 \ \ W/m \cdot K$$
$$Pr = 1081$$
$$Re = 929$$

(a)

$$\delta_V = \frac{5x}{\sqrt{Re}} = 0.1312 \ \ m$$

(b)

$$\delta_T = \frac{\delta}{(Pr)^{\frac{1}{3}}} = 0.01278 \quad m$$

(c)
For laminar flow:

$$Nu_x = \frac{hx}{k} = 0.332\left(Re^{1/2}\right)\left(Pr^{1/3}\right)$$

h = Heat transfer coefficient (W/m²·K)
k = Thermal conductivity of the material (W/m·K)

$$h = \frac{k}{x} 0.332\left(Re^{1/2}\right)\left(Pr^{1/3}\right) = 18.17 \quad (\mathrm{W/m^2 \cdot K})$$

Heat flux:

$$q_x = q_w = h(T_s - T_\infty) = (18.17)(20 - 100) = -1453.6 \quad \mathrm{W/m^2}$$

(d)

From Blasius, the local shear stress:

$$\tau = C_d \cdot \frac{1}{2}\rho U^2 = \frac{0.6642}{\sqrt{Re}} \cdot \frac{1}{2}\rho U^2 = 0.094 \quad N/m^2$$

(e)

Total drag force per unit width, for 2 surfaces:

$$F_D = 2C_D \cdot \frac{1}{2}\rho(L\times W)U^2 = \frac{1.328}{\sqrt{Re}} \cdot \frac{1}{2}\rho(L\times W)U^2$$

$$\frac{F_D}{W} = 2C_D \cdot \frac{1}{2}\rho(L)U^2 = \frac{1.328}{\sqrt{Re}} \cdot \rho(L)U^2 = 0.301 \ \ N/m$$

(f)

The magnitude of the heat transfered per unit width of the plate (W/m), (2 plates):

$$Q = hA(T_s - T_\infty) = h(L\times W)(T_s - T_\infty)$$

$$\frac{Q}{W} = 2h(L)(T_s - T_\infty) = -4651.5 \quad W/m$$

HEAT FLUX

Heat flux or thermal flux, q_w is defined as the amount of heat transferred per unit area per unit time from or to a surface. In SI its units are watts per square metre (W/m²).

$$q_w = \frac{d}{dx}\left[\int_0^{\infty} \rho c_p u(T - T_e)\right] dy$$

Known that:

$$\frac{T - T_e}{T_w - T_e} = \left(1 - \frac{u}{U}\right)$$

$$\frac{u}{U} = \frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

If the laminar flow velocity profile is assumed to be quadratic, we can evaluate the heat transferred approximately by using above mentioned equations:

We could write above equations as:

$$u = U\left(\frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2\right) \quad [1]$$

$$T - T_e = (T_w - T_e)\left(1 - \frac{u}{U}\right) \quad [2]$$

$$q_w = \frac{d}{dx}\left[\int_0^{\infty} \rho c_p u (T - T_e)\right] dy \quad [3]$$

From [1] and [2], [3] can be written as:

$$\begin{aligned}
q_w &= \frac{d}{dx}\left[\int_0^{\infty} \rho c_p u (T - T_e)\right] dy \\
&= \frac{d}{dx}\left[\int_0^{\infty} \rho c_p U\left(\frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)(T_w - T_e)\left(1 - \frac{u}{U}\right) dy\right] \\
&= \frac{d}{dx}\left[\rho c_p U (T_w - T_e)\int_0^{\infty}\left(\frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)\left(1 - \frac{u}{U}\right) dy\right] \\
&= \frac{d}{dx}\left[\rho c_p U (T_w - T_e)\int_0^{\delta_T}\left(\frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)\left(1 - \left(\frac{2y}{\delta_T} - \left(\frac{y}{\delta_T}\right)^2\right)\right) dy\right]
\end{aligned}$$

$$q_w = \frac{d}{dx}\left[\rho c_p U(T_w - T_e)\int_0^{\delta_T}\left(\frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)\left(1 - \frac{2y}{\delta_T} + \left(\frac{y}{\delta_T}\right)^2\right)dy\right]$$

We solve the integral part first:

$$\int_0^{\delta_T}\left(\frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)\left(1 - \frac{2y}{\delta_T} + \left(\frac{y}{\delta_T}\right)^2\right)dy = \int_0^{\delta_T}\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{2y}{\delta_T} + \frac{y^2}{\delta_T^2}\right)dy$$

$$= \frac{\delta_T^2}{\delta}\left(\frac{1}{6}\right) - \frac{\delta_T^3}{\delta^2}\left(\frac{1}{30}\right)$$

Multiply with $\frac{\delta}{\delta}$ $$= \delta\left[\frac{\delta_T^2}{\delta^2}\left(\frac{1}{6}\right) - \frac{\delta_T^3}{\delta^3}\left(\frac{1}{30}\right)\right]$$

Assumed $\frac{\delta_T}{\delta} = B$ $$= \delta\left[\frac{B^2}{6} - \frac{B^3}{30}\right]$$

Heat flux will become:

$$
\begin{aligned}
q_w &= \frac{d}{dx}\left[\rho c_p U(T_w - T_e)\int_0^{\delta_T}\left(\frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)\left(1 - \frac{2y}{\delta_T} + \left(\frac{y}{\delta_T}\right)^2\right)dy\right] \\
&= \frac{d}{dx}\left[\rho c_p U(T_w - T_e)\delta\left[\frac{B^2}{6} - \frac{B^3}{30}\right]\right]
\end{aligned}
$$

From basic equation:

$$
T - T_e = (T_w - T_e)\left(1 - \frac{2y}{\delta_T} + \frac{y^2}{\delta_T^2}\right)
$$

Differentiate T to y

$$
\frac{dT}{dy} = (T_w - T_e)\left(-\frac{2}{\delta_T} + \frac{2y}{\delta_T^2}\right)
$$

$$
\left(\frac{dT}{dy}\right)_{y=0} = (T_w - T_e)\left(-\frac{2}{\delta_T}\right)
$$

Known that:

$$q_w = (T_w - T_e)\left(1 - \frac{2y}{\delta_T} + \frac{y^2}{\delta_T^2}\right)$$

Differentiate T to y

$$\frac{dT}{dy} = -k\left(\frac{dT}{dy}\right)_{y=0}$$

$$= -k(T_w - T_e)\left(-\frac{2}{\delta_T}\right) = \frac{2k(T_w - T_e)}{\delta_T}$$

$$= \frac{2k(T_w - T_e)}{B\delta}$$

$B = \frac{\delta_T}{\delta}$, $k = \rho c_p U$