CHAPTER 4
LAMINAR BOUNDARY LAYER

INVISCID FLOW PAST WEDGES AND CORNERS
FALKNER-SKAN SOLUTION

In fluid dynamics, the Falkner–Skan boundary layer (named after V. M. Falkner and Sylvia W. Skan) describes the steady two-dimensional laminar boundary layer that forms on a wedge, i.e. flows in which the plate is not parallel to the flow. It is a generalization of the Blasius boundary layer.

Falkner and Skan (1931) found that similarity was achieved by the variable $\eta = Cyx^a$, which is consistent with a power-law free stream velocity distribution.

$$U(x) = Kx^m$$

$$m = 2a + 1$$

The exponent m may be termed the Falkner-Skan power-law parameter. The constant C must make η dimensionless but is otherwise arbitrary.

The best choice is $C^2 = \frac{K(1+m)}{2v}$, which is consistent with its limiting case for $m = 0$, the Blasius variable.

Thus;

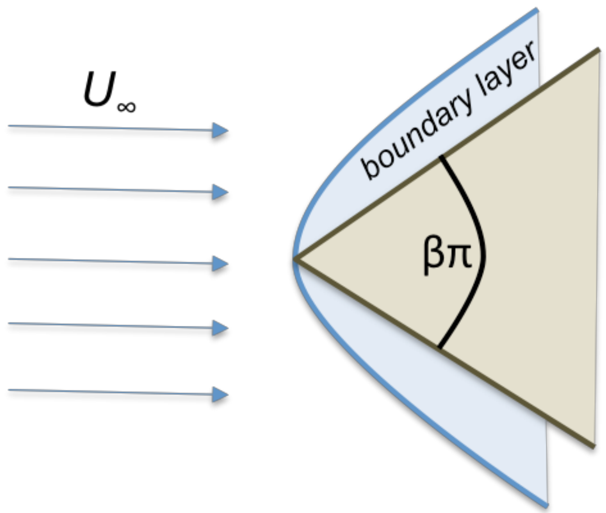
$$\eta = y\sqrt{\frac{m+1}{2}\frac{U(x)}{\upsilon x}} = y\sqrt{\frac{m+1}{2}\frac{U}{\upsilon x}}$$

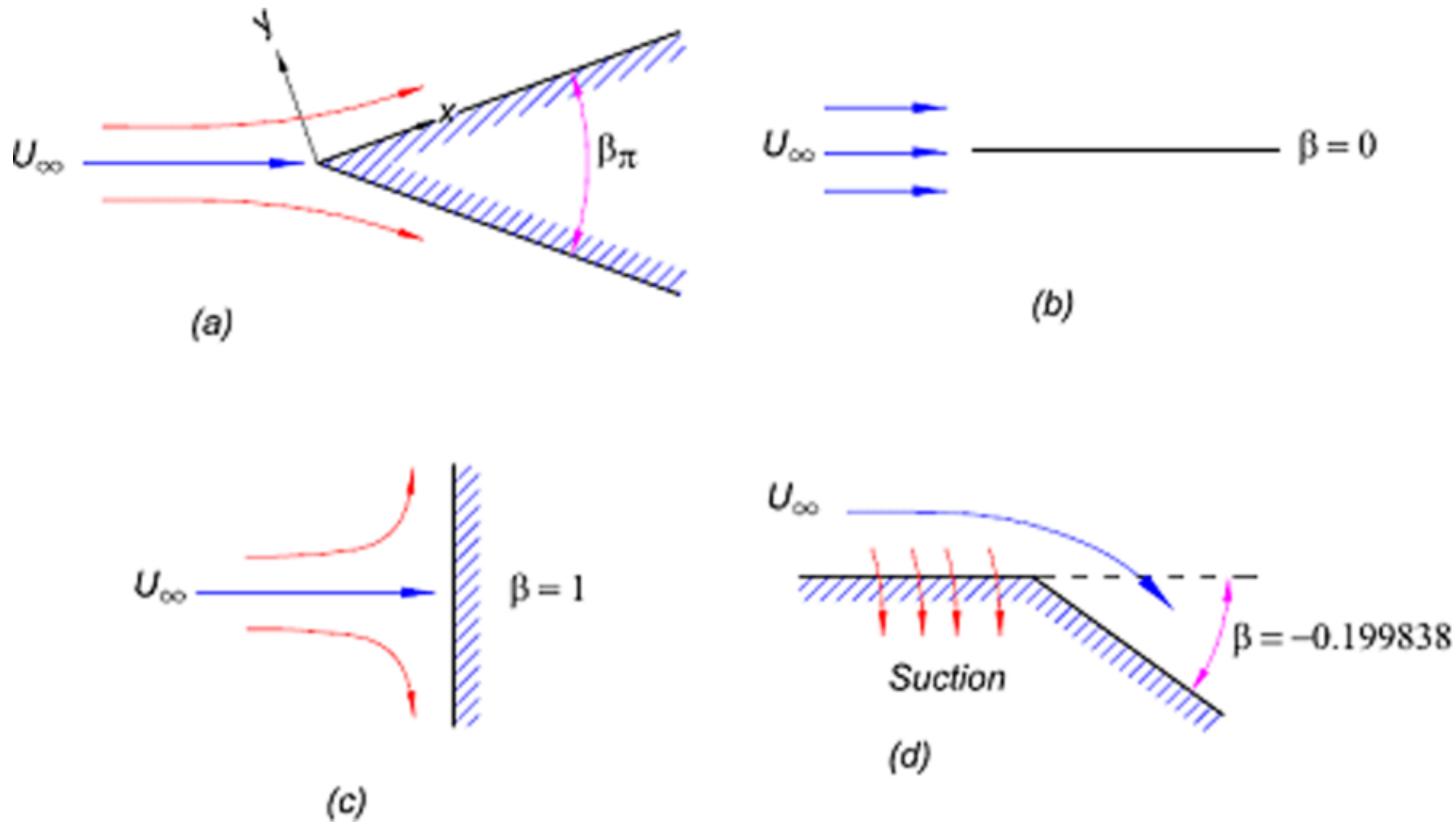
Substituting this particular C into Eq.4-67, gives the most common form of the Falkner-Skan equation for similar flows:

$$f''' + ff'' + \beta(1 - f'^2) = 0$$

where

$$\beta = \frac{2m}{1+m}$$





Y
x
U_∞
$\beta\pi$
(a)
U_∞
$\beta = 0$
(b)
U_∞
$\beta = 1$
(c)
U_∞
$\beta = -0.199838$
Suction
(d)

EXAMPLE

Estimate the variation of surface velocity along the wall if angle of wedge is 10°.

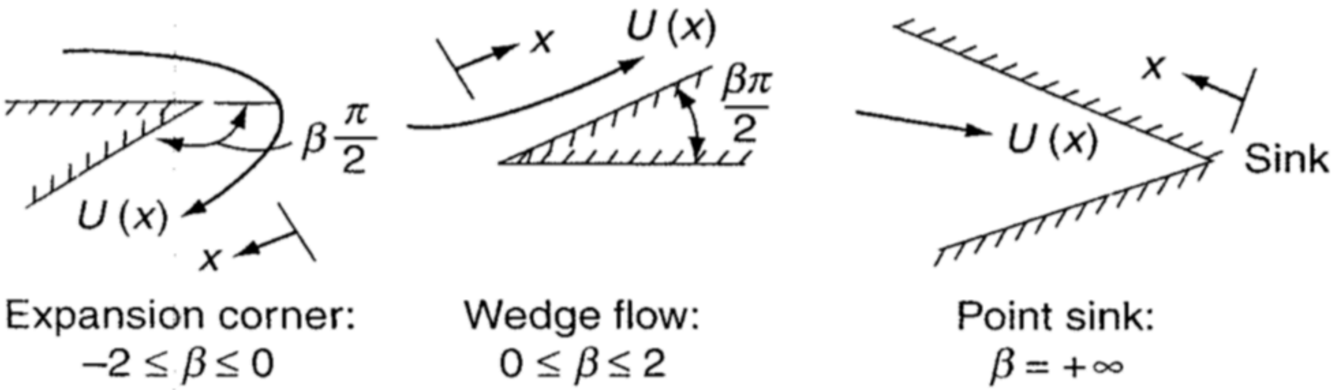


FIGURE 4-10

Some examples of Falkner–Skan potential flows.

Known that:

$U(x) = Kx^m$ and $m = 2a + 1$

$\frac{\beta\pi}{2} = 10°$, so that $\beta = \frac{1}{9}$

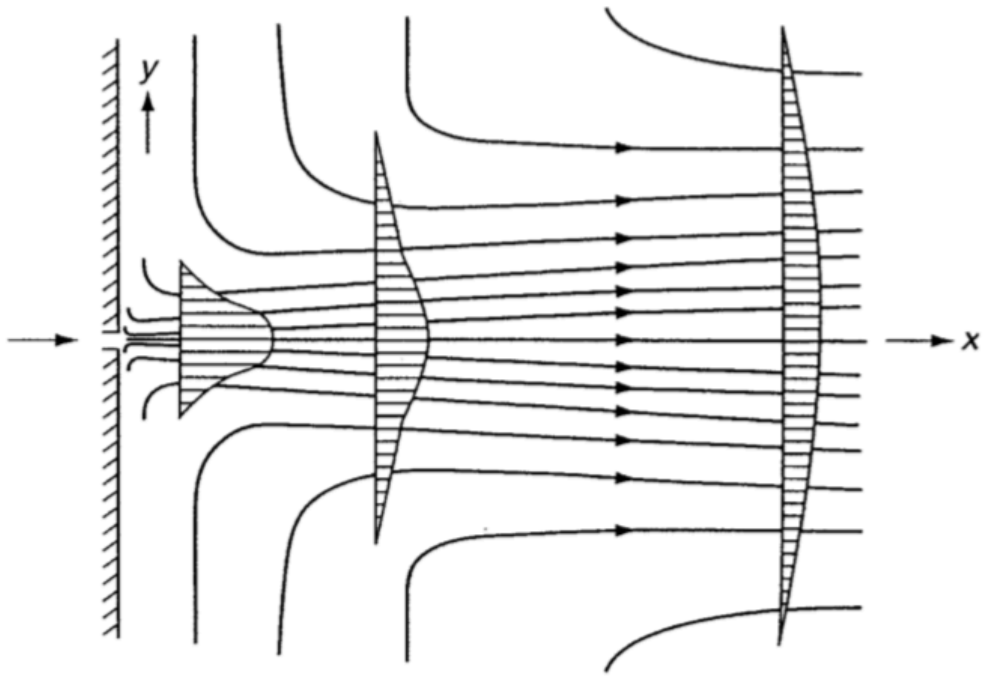
$\beta = \frac{2m}{1+m} = \frac{1}{9}$, so that $m = \frac{1}{17}$

Velocity profile is $U(x) = Kx^{\frac{1}{17}}$

β	m	Description of flow
$-2 \leq \beta \leq 0$	$-\frac{1}{2} \leq m \leq 0$	Flow around an expansion corner of turning angle $\frac{\beta\pi}{2}$
$\beta = 0$	$m = 0$	The flat plate
$0 \leq \beta \leq +2$	$0 \leq m \leq \infty$	Flow against a wedge of half angle $\frac{\beta\pi}{2}$
$\beta = 1$	$m = 1$	The plane stagnation point ($180°$ wedge)
$\beta = +4$	$m = -2$	Doublet flow near a plane wall
$\beta = +5$	$m = -\frac{5}{3}$	Double flow near a $90°$ corner
$\beta = +\infty$	$m = -1$	Flow toward a point sink

THE PLANE LAMINAR JET FLOW

Consider a plane jet emerging into a still ambient fluid from a slot at $x = 0$, as shown below:



In this situation, the conservation of momentum was applied.

$$\text{Momentum flow,} \qquad \dot{m}V = \rho(V \cdot n)AV = \rho V_n AV$$

The momentum flux is defined as the momentum flow per unit area.

We could simplify it as:

$$\text{Momentum flux,}\quad J = \rho \int_{-\infty}^{+\infty} u^2\, dy = \frac{16}{9}\rho\left(v^{1/2}\right)a^3$$

$$a = constant$$

The maximum velocity can be concluded as:

$$u_{max} = 0.4543\left(\frac{J^2}{\rho\mu x}\right)^{1/3}$$

We may define the width of the jet as twice the distance y where $u = 0.01u_{max}$:

$$Width = (2y)_{1\%} = b = 21.8\left(\frac{x^2\mu^2}{J\rho}\right)^{1/3}$$

The mass flow rate across any vertical plane is given by:

$$\dot{m} = \rho \int_{-\infty}^{+\infty} u\, dy = (36J\rho\mu x)^{1/3} = 3.302(J\rho\mu x)^{1/3}$$

which is seen to increase with $x^{1/3}$ as the jet entrains ambient fluid by dragging it along.

This result is correct at large x but implies falsely that $\dot{m} = 0$ at $x = 0$, which is the slot where the jet issues.

The reason is that the boundary layer approximations fail if the Reynolds number is small, and the appropriate Reynolds number here is :

$$Re = \frac{\dot{m}}{\mu}$$

Thus, the solution is invalid for small values of Reynolds number of :

$$Re = \frac{J\rho x}{\mu^2}$$

EFFECT OF PRESSURE GRADIENT
SEPARATION AND FLOW OVER CURVED SURFACES

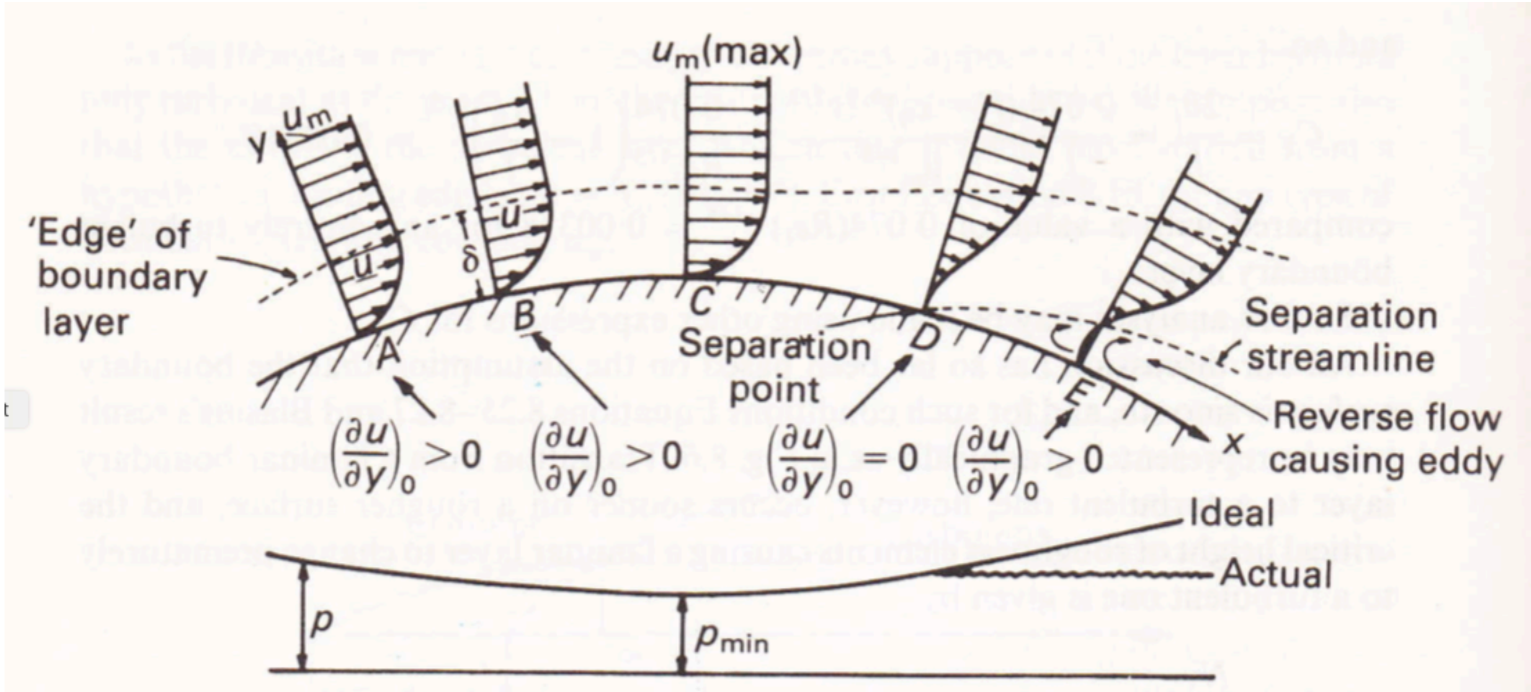
All solid objects traveling through a fluid (or alternatively a stationary object exposed to a moving fluid) acquire a boundary layer of fluid around them where viscous forces occur in the layer of fluid close to the solid surface. Boundary layers can be either laminar or turbulent. A reasonable assessment of whether the boundary layer will be laminar or turbulent can be made by calculating the Reynolds number of the local flow conditions.

Flow separation occurs when the boundary layer travels far enough against an adverse pressure gradient that the speed of the boundary layer relative to the object falls almost to zero. The fluid flow becomes detached from the surface of the object, and instead takes the forms of eddies and vortices.

Boundary layer separation is the detachment of a boundary layer from the surface into a broader wake. Boundary layer separation occurs when the portion of the boundary layer closest to the wall or leading edge reverses in flow direction. The separation point is defined as the point between the forward and backward flow, where the shear stress is zero. The overall boundary layer initially thickens suddenly at the separation point and is then forced off the surface by the reversed flow at its bottom.

We have so far considered flow in which the pressure outside the boundary layer is constant. If, however, the pressure varies in the direction of flow, the behaviour of the fluid may be greatly affected.

Let us consider flow over a curved surface as illustrated below. The radius of curvature is everywhere large compared with the boundary layer thickness.



As the fluid is deflected round the surface, it is accelerated over the left-hand section until at position C, the velocity just outside the boundary layer is a maximum. Here, the pressure is a minimum.

Thus, from A to C, the pressure gradient $\frac{\partial p}{\partial x}$ is negative.

The net pressure force on an element in the boundary layer is in the forward direction.

Beyond point C, however, the pressure increases, and so the net pressure force on an element in the boundary layer opposes the forward flow.

At point D, the value of $\frac{\partial u}{\partial y}$ at the surface is zero.

Further downstream, at point E, the flow close to the surface has actually been reversed. The fluid no longer able to follow the contour of the surface, breaks away from it.

This breakaway before the end of the surface is reached is usually termed separation.

It first occurs at the separation point where $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ become zero.

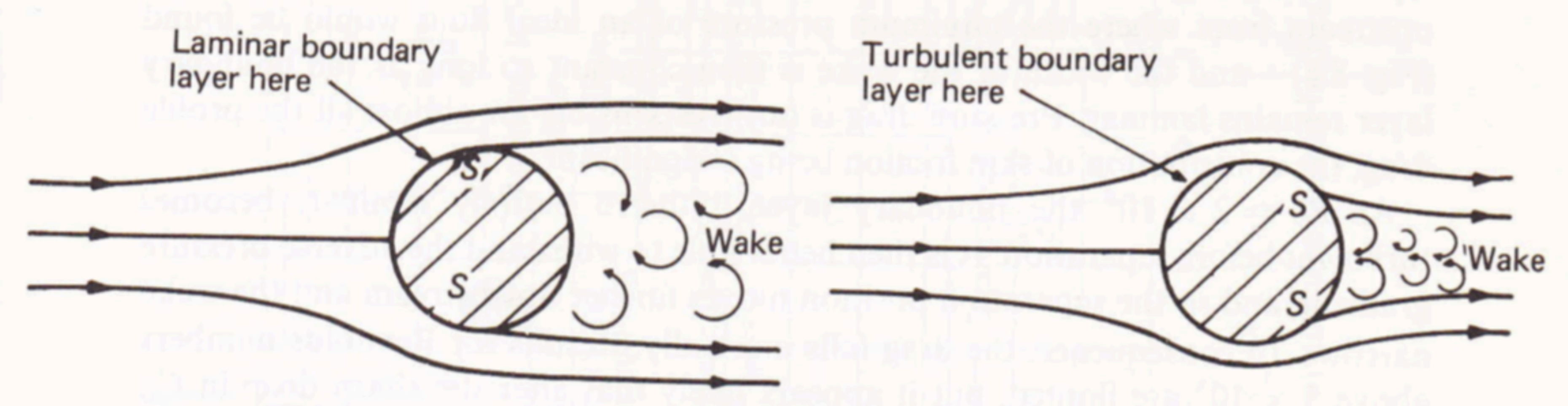
It is caused by the reduction of velocity in the boundary layer, combined with a positive pressure gradient.

Separation can therefore occur only when an adverse (positive) pressure gradient exists. Flow over a flat plate with zero or negative pressure gradient will never separate before reaching the end of the plate, no matter how long the plate.

In an ideal fluid, separation from a continuous surface would never occur, even with an adverse pressure gradient because there would be no friction to produce a boundary layer along the surface.

The line of zero velocity dividing the forward and reverse flow leaves the surface at the separation point, and is known as the separation streamline. As a result of the reverse flow, large irregular eddies are formed in which much energy is dissipated as heat.

Separation occurs with both laminar and turbulent boundary layer. However, laminar boundary layers are much more prone to separation that turbulent ones. This is because in a laminar boundary layer, the increase of velocity with distance from the surface is less rapid, and the adverse pressure gradient can more readily halt the slow-moving fluid close to the surface.



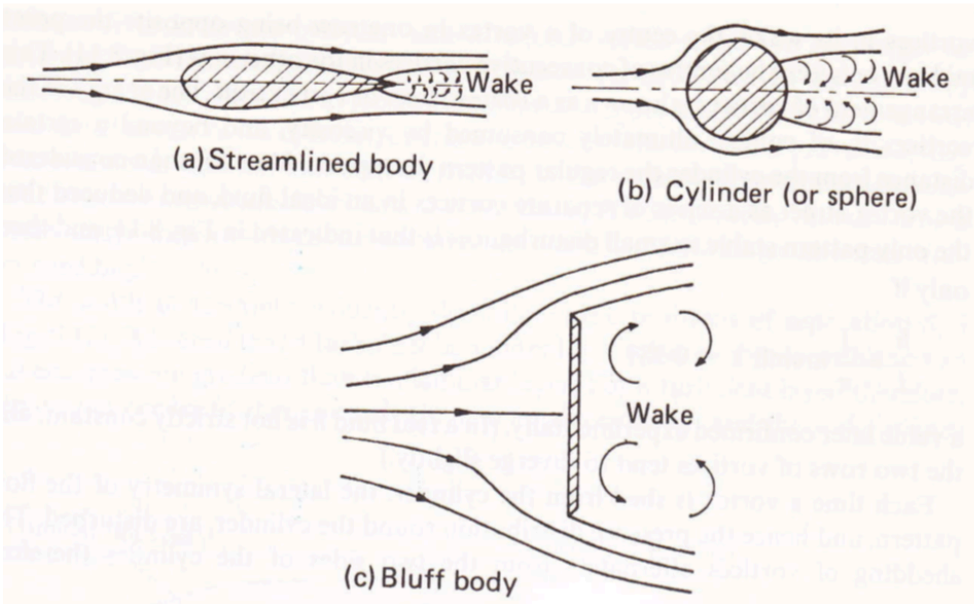
A turbulent boundary layer can survive an adverse pressure gradient for some distance before separating.

For any boundary layer, however, the greater the adverse pressure gradient, the sooner separation occurs. The boundary layer thickens rapidly in an adverse pressure gradient, and the assumption that δ is small may no longer be valid.

In almost all cases in which flow takes place round a solid body, the boundary layer separates from the surface at some point. One exception is an infinitesimally thin flat plate parallel to the main stream. Downstream of the separation position the flow is greatly disturbed by large-scale eddies, and this region of eddying motion is usually known as the wake.

As a result of the energy dissipated by the highly turbulent motion in the wake, the pressure there reduced and the pressure drag on the body is thus increased. The magnitude of the pressure drag depends very much on the size of the wake and this, in turn, depends on the position of separation.

If the shape of the body is such that separation occurs only well towards the rear, and the wake is small, the pressure drag is also small. Such a body is termed a streamlined body. For bluff body, on the other hand, the flow is separated over much of the surface, the wake is large and the pressure drag is much greater than the skin friction.



DEVELOPMENT OF WAKE BEHIND CYLINDER

The flow pattern in the wake depends on the Reynolds number of the flow.

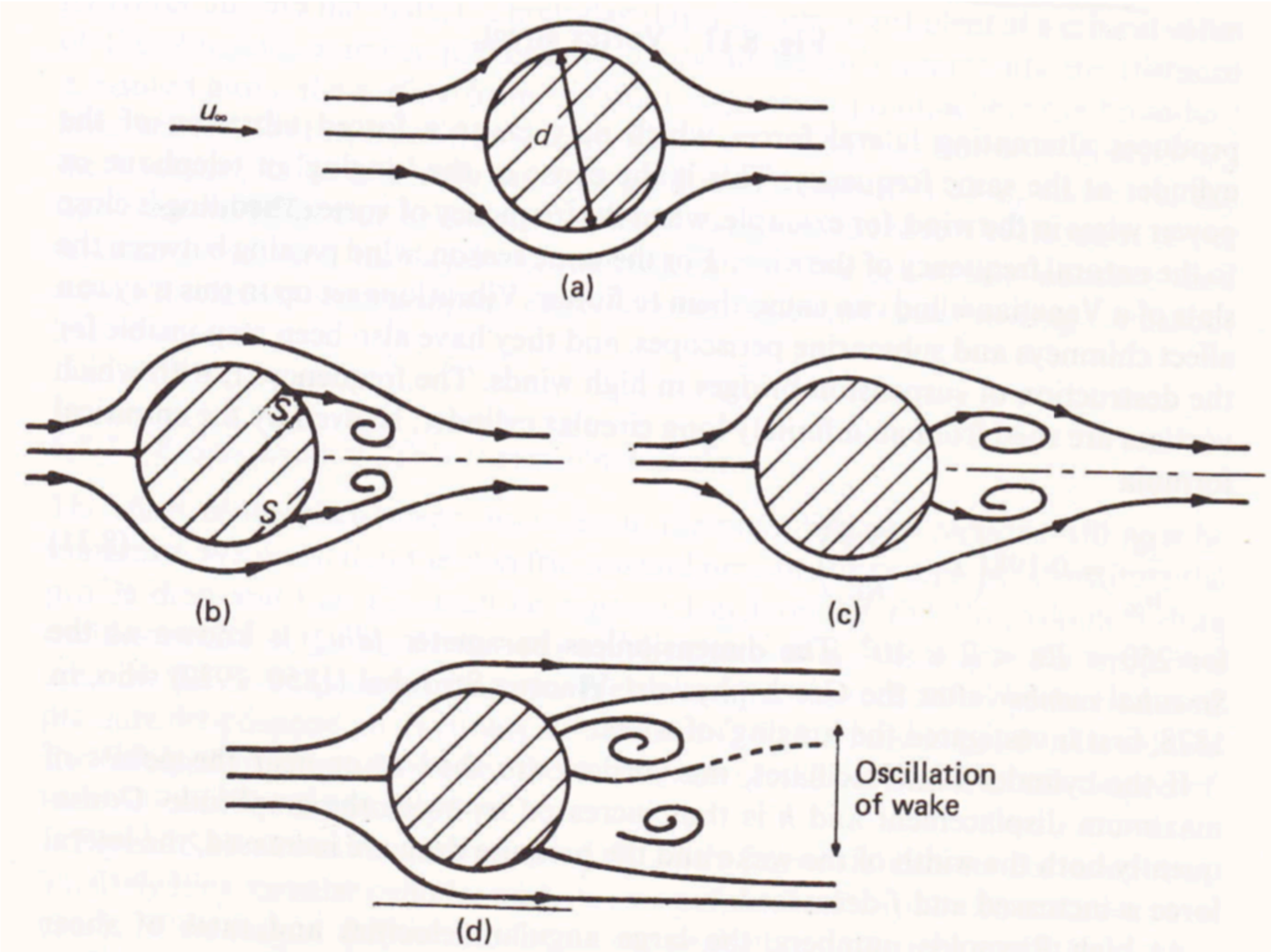
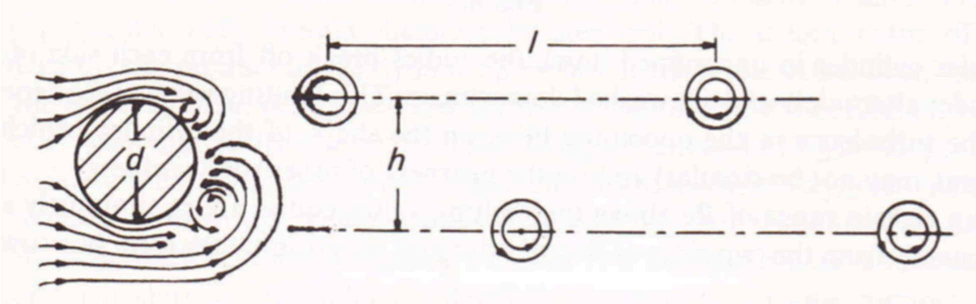


Figure (a):
For very low Reynolds number ($Re < 0.5$), the inertia forces are negligible, and the streamlines come together behind the cylinder.

Figure (b):
If Reynolds number increased to the range 2-30, the boundary layer separates symmetrically from the two sides at the positions S. Two eddies are formed which rotate in opposite directions. At these Reynolds number, they remain unchanged in position, their energy being maintain by the flow from the separated boundary layer. Behind the eddies, however, the main streamlines come together. The length of the wake is limited.

Figure (c):
With increase of Reynolds number, the eddies was elongated but the arrangement is unstable.

Figure (d):
At Reynolds number 40-70, for a circular cylinder, a periodic oscillation of the wake is observed. Then, at a certain limiting value of Reynolds number, usually about 90 for a circular cylinder in unconfined flow, the eddies break off from each side of the cylinder alternately and are washed downstream. This limiting value of Reynolds number depends on the turbulence of the oncoming flow, on the shape of the cylinder and on the nearness of other solid surfaces.



In a certain range of Reynolds number above the limiting value, eddies are continuously shed alternately from the two sides of the cylinder and, as a result, they form two rows of vortices in its wake, the centre of a vortex in one row being opposite the point midway between the centres of consecutive vortices in the other row.

This arrangement of vortices is known as a vortex street or vortex trail.

Von Karman considered the vortex street as a series of separate vortices in an ideal fluid and deduced that the only pattern stable to small disturbance, and then only if:

$$\frac{h}{l} = \frac{1}{\pi} arcsinh\, 1 = 0.281$$

A value later is confirmed experimentally.

WAKE OF AN AIRFOIL

A wake is the defect is stream velocity behind an immersed body in a flow, a shown below.

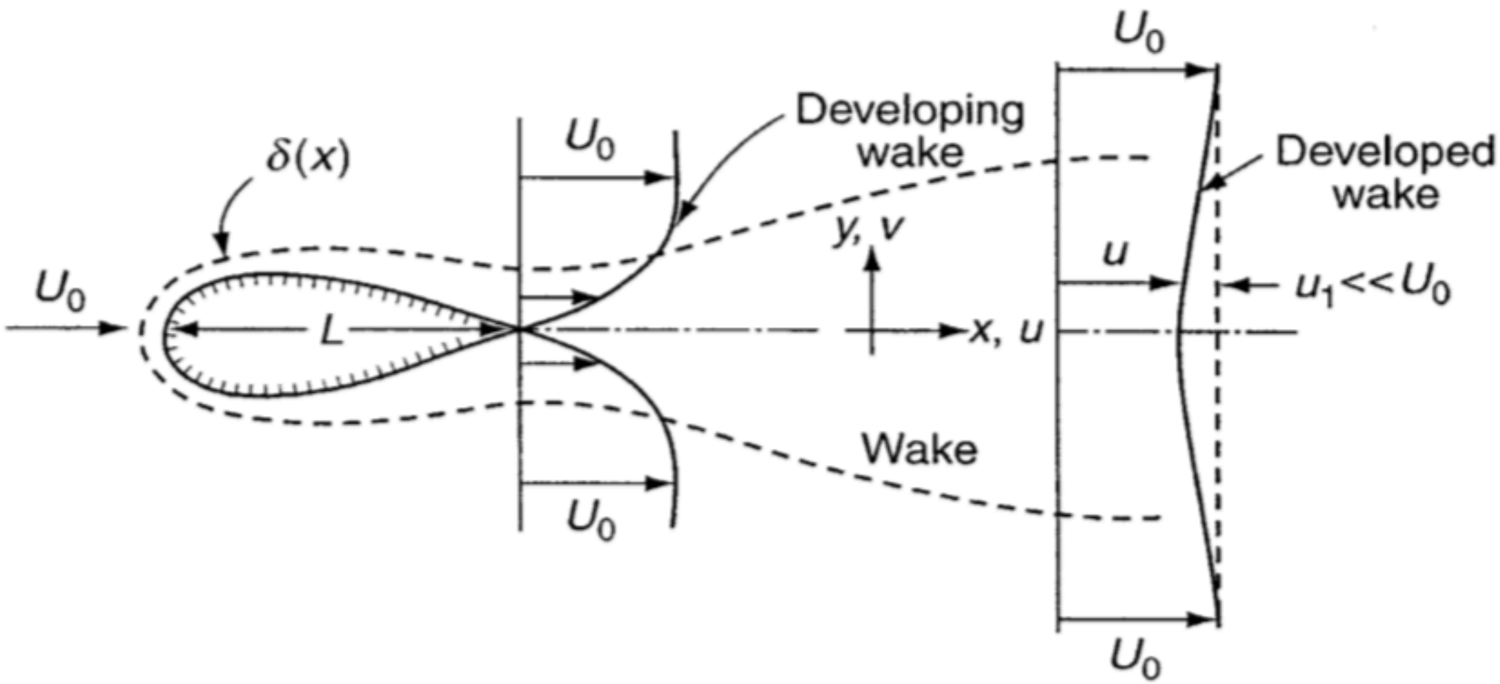


FIGURE 4-19
Flow in the wake of a body immersed in a stream.

Drag force can be written as:

$$F_D = C_D \cdot \frac{1}{2}\rho A U^2$$

The wake velocity may be written as:

$$\frac{u_1}{U_0} = C_D \left(\frac{Re_L}{16\pi}\right)^{1/2} \left(\frac{L}{x}\right)^{1/2} exp\left(-\frac{U_0 y^2}{4x\upsilon}\right)$$

u_1 = Defect velocity
U_0 = Free stream velocity
υ = Kinematic viscosity of fluid

THE CORRELATION METHOD OF THWAITES

From Von Karman integral relation, we can rewrite the momentum relation in the more compact form as:

$$\frac{\tau_w}{\rho U^2} = \frac{C_f}{2} = \frac{d\theta}{dx} + (2 + H)\frac{\theta}{U}\frac{dU}{dx} \qquad \text{(Eq.4-122)}$$

$$H = \frac{\delta^*}{\theta}$$

$$\lambda = \frac{\theta^2}{\upsilon} \cdot \frac{dU}{dx}$$

$$\upsilon = \text{Kinematic viscosity}$$

Multiply the momentum-integral relation by $\frac{U\theta}{\upsilon}$

$$\left(\frac{U\theta}{\upsilon}\right)\frac{\tau_w}{\rho U^2} = \left(\frac{U\theta}{\upsilon}\right)\frac{d\theta}{dx} + \left(\frac{U\theta}{\upsilon}\right)(2+H)\frac{\theta}{U}\frac{dU}{dx}$$

$$\frac{\tau_w \theta}{\mu U} = \left(\frac{U\theta}{\upsilon}\right)\frac{d\theta}{dx} + \left(\frac{\theta^2}{\upsilon}\right)\frac{dU}{dx}(2+H) \qquad \text{(Eq.4-133)}$$

Now, H and the left-hand side of this equation are dimensionless boundary layer function. Thus, by assumption, are correlated reasonably by a single parameter (λ in this case). Thus we assume, after Holstein and Bohlen (1940), that:

$$\frac{\tau_w \theta}{\mu U} \approx S(\lambda) \qquad \text{Shear correlation}$$

$$H = \frac{\delta^*}{\theta} \approx H(\lambda) \qquad \text{Shape-factor correlation}$$

And further note that:

$$\theta d\theta = d\left(\frac{\theta^2}{2}\right)$$

Eq.4-133 may thus be rewritten as:

$$U\frac{d}{dx}\left(\frac{\lambda}{U'}\right) \approx 2[S(\lambda) - \lambda(2+H)] = F(\lambda) \qquad \text{(Eq.4-135)}$$

Whereas earlier workers would have proposed a family of profiles to evaluate the parametric functions in Eq.4-135, Thwaites (1949) abandoned the favorite-family idea and looked at the entire collection of known analytic and experimental results to see if they could be fit by a set of average one-parameter functions. As shown in Figure 4-22, he found excellent correlation for the function $F(\lambda)$ and proposed a simple linear fit.

$$F(\lambda) \approx 0.45 - 6.0\lambda \qquad \text{(Eq.4-136)}$$

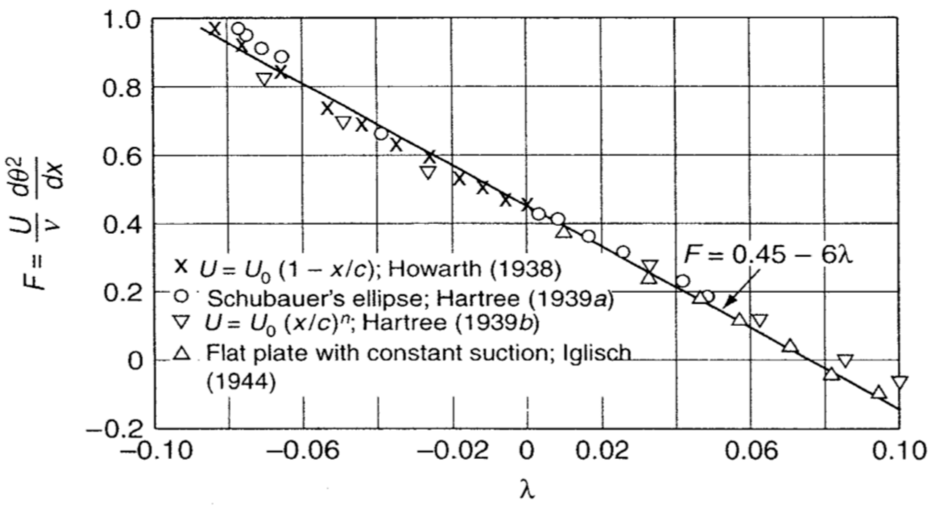


FIGURE 4-22
Empirical correlation of the boundary-layer function $F(\lambda)$ in Eq. (4-135). [*After Thwaites (1949)*.]

If $F = a - b\lambda$, Eq.4-135 has a closed-form solution which the reader may verify as an exercise:

$$\frac{\theta^2}{\upsilon} = aU^{-b}\left(\int_{x_0}^{x} U^{b-1} \cdot dx + C\right) \qquad \text{(Eq.4-137)}$$

If x_0 is a stagnation point, the constant C mest be zero to avoid an infinite momentum thickness where $U = 0$. Thus, Thwaites has shown that $\theta(x)$ is predicted very accurately ($\pm 3\%$), for all types of laminar boundary layers, by the simple quadrature.

$$\theta^2 \approx \frac{0.45\upsilon}{U^6}\int_0^x U^5 \cdot dx \qquad \text{(Eq.4-138)}$$

Thwaites' suggested correlations for $S(\lambda)$ and $H(\lambda)$ as follows:

$$S(\lambda) \approx (\lambda + 0.09)^{0.62} \qquad \text{(Eq.4-140)}$$

$$H(\lambda) \approx 2.0 + 4.14z - 83.5z^2 + 854z^3 - 3337z^4 + 4576z^5 \qquad \text{(Eq.4-141)}$$

$$z = 0.25 - \lambda$$