BOUNDARY LAYER

TURBULENT BOUNDARY LAYER
EMPIRICAL FORM

The empirical method of predicting turbulent flow quantities on a flat plate with zero pressure gradient is based entirely on data.

It is more accurate than the power-law form but also more complicated.

The time average turbulent velocity profile can be divided into two regions, the inner region and the outer region.

The velocity profile for inner region is defined as :

$$\frac{u}{u_\tau} = f\left(\frac{u_\tau y}{\nu}\right)$$

$$u_\tau = \sqrt{\frac{\tau_0}{\rho}} = \text{ shear velocity}$$

The velocity profile for outer region is defined as :

$$\frac{U_\infty - \bar{u}}{u_\tau} = f\left(\frac{y}{\delta}\right)$$

$$U_\infty - \bar{u} = \text{velocity defect}$$

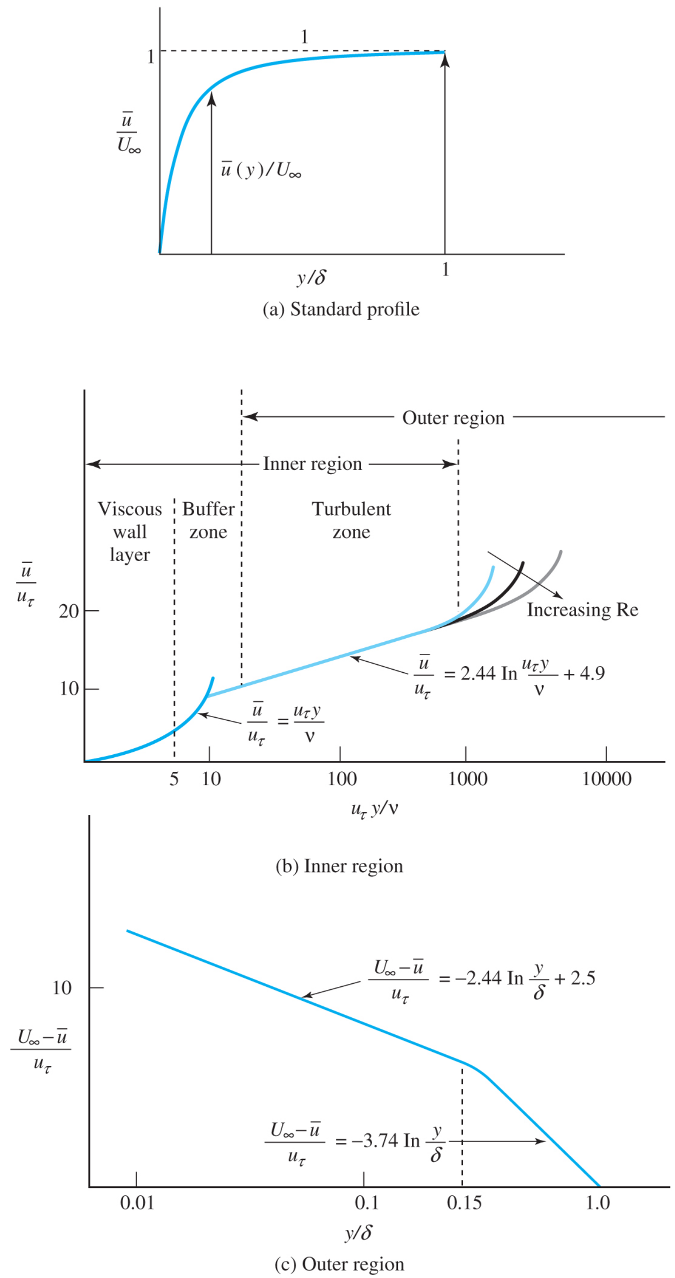

Figure 8.26 Velocity profile in a turbulent boundary layer.

The equations above involve the shear velocity, u_τ, which depends on the wall shear stress, τ_0. There are several such relationships used; one that gives excellent results is:

$$C_f = C_d = \frac{0.455}{[\ln(0.06Re_x)]^2}$$

C_f = local skin friction coefficient

This local skin friction coefficient (local drag coefficient) allows us to determine τ_0 and thus u_τ at any location of interest. The velocity profiles can be used to calculated quantities of interest but u_τ must be known.

Assuming turbulent flow the leading edge, the shear stress can be integrated to yield the drag. Then the skin friction coefficient (drag coefficient) becomes:

$$C_D = \frac{0.523}{[\ln(0.06Re_x)]^2}$$

This relation is very good and can be used up to

$Re = 10^9$ with an error of 2% or less.

Even at $Re = 10^{10}$ the error is about 4%.

Finally, this equation can be summarized:

$$\frac{U_\infty}{u_\tau} = 2.44 \cdot \ln \frac{u_\tau \cdot \delta}{\nu} + 7.4$$

this equation allows an easy calculation of δ by knowing u_τ.

EXAMPLE 1 :

Estimate the boundary layer thickness at the end of a 4.5 m long flat plate if wind flow on it at 30 m/s. Also calculate the drag force on one side if the plate is 3 m wide. Use the empirical data. Assume that the temperature is 20°C.

Properties of Air at Atmospheric Pressure

Temperature T (°C)	*Density* ρ (kg/m^3)	*Viscosity* μ (N·s/m^2)	*Kinematic viscosity* ν (m^2/s)	*Velocity of sound* c (m/s)
−50	1.582	1.46×10^{-5}	0.921×10^{-5}	299
−30	1.452	1.56	1.08×10^{-5}	312
−20	1.394	1.61	1.16	319
−10	1.342	1.67	1.24	325
0	1.292	1.72	1.33	331
10	1.247	1.76	1.42	337
20	1.204	1.81	1.51	343
30	1.164	1.86	1.60	349
40	1.127	1.91	1.69	355
50	1.092	1.95	1.79	360
60	1.060	2.00	1.89	366
70	1.030	2.05	1.99	371
80	1.000	2.09	2.09	377
90	0.973	2.13	2.19	382
100	0.946	2.17	2.30	387
200	0.746	2.57	3.45	436
300	0.616	2.93×10^{-5}	4.75×10^{-5}	480

ANSWER 1 :

The boundary layer thickness can be calculated by using this equation:

$$\frac{U_\infty}{u_\tau} = 2.44 \cdot \ln \frac{u_\tau \cdot \delta}{\nu} + 7.4$$

However, we need to define the value of shear velocity, u_τ :

$$u_\tau = \sqrt{\frac{\tau_0}{\rho}} \quad \text{and} \quad \tau_0 = C_d \cdot \frac{1}{2}\rho U^2 \quad \text{and} \quad C_d = \frac{0.455}{(\ln(0.06Re_x))^2}$$

$$Re_x = \frac{\rho U x}{\mu} = \frac{Ux}{\nu} = \frac{(30)(4.5)}{1.51 \times 10^{-5}} = 8{,}940{,}397$$

$$\ln(0.06Re_x) = \ln(0.06 \times 8{,}940{,}397) = 13.1926$$

$$\tau_0 = \left(\frac{0.455}{(\ln(0.06Re_x))^2}\right) \cdot \frac{1}{2}\rho U^2 = 1.416 \;\; Pa$$

$$u_\tau = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{1.416}{1.204}} = 1.084 \;\; m/s$$

$$\frac{U_\infty}{u_\tau} = 2.44 \cdot \ln\frac{u_\tau \cdot \delta}{\nu} + 7.4$$

$$\frac{30}{1.084} = 2.44 \cdot \ln\frac{(1.084) \cdot \delta}{1.51 \times 10^{-5}} + 7.4$$

$$\delta = 0.05567 \;\; m$$

Drag force can be calculated by using this equation:

$$C_D = \frac{0.523}{(\ln(0.06Re_L))^2} = \frac{0.523}{13.1926^2} = 0.003$$

$$F_D = C_D \frac{1}{2} \rho A U^2 = (0.003) \frac{1}{2} (1.204)(4.5 \times 3)(30)^2 = 21.94 \ \ N$$

EXAMPLE 2 :

Estimate the maximum boundary layer thickness and the drag due to friction on the side of a ship that measures 40 m long with a submerged depth of 8 m assuming the side of the ship is approximated as a flat plate. The ship travels at 10 m/s. Use the empirical methods and compare with the results using the power-law model.

Properties of Water

Temperature T (°C)	*Density* ρ (kg/m³)	*Viscosity* μ (N·s/m²)	*Kinematic viscosity* ν (m²/s)	*Surface tension* σ (N/m)	*Vapor pressure* p_v (kPa)	*Bulk modulus* B (Pa)
0	999.9	1.792×10^{-3}	1.792×10^{-6}	0.0762	0.610	204×10^{7}
5	1000.0	1.519	1.519	0.0754	0.872	206
10	999.7	1.308	1.308	0.0748	1.13	211
15	999.1	1.140	1.141	0.0741	1.60	214
20	998.2	1.005	1.007	0.0736	2.34	220
30	995.7	0.801	0.804	0.0718	4.24	223
40	992.2	0.656	0.661	0.0701	3.38	227
50	988.1	0.549	0.556	0.0682	12.3	230
60	983.2	0.469	0.477	0.0668	19.9	228
70	977.8	0.406	0.415	0.0650	31.2	225
80	971.8	0.357	0.367	0.0630	47.3	221
90	965.3	0.317	0.328	0.0612	70.1	216
100	958.4	0.284×10^{-3}	0.296×10^{-6}	0.0594	101.3	207×10^{7}

ANSWER 2 :

We take properties of water at 5°C.

$$\tau_0 = \left(\frac{0.455}{(\ln(0.06Re_x))^2}\right) \cdot \frac{1}{2}\rho U^2 = 82.80 \;\; Pa$$

$$u_\tau = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{82.80}{1000}} = 0.288 \;\; m/s$$

$$\frac{U_\infty}{u_\tau} = 2.44 \cdot \ln\frac{u_\tau \cdot \delta}{\nu} + 7.4$$

$$\frac{10}{0.288} = 2.44 \cdot \ln\frac{(0.288)\cdot\delta}{1.519 \times 10^{-6}} + 7.4$$

$$\delta = 0.384 \;\; m$$

$$C_D = \frac{0.523}{(\ln(0.06Re_L))^2} = \frac{0.523}{16.5755^2} = 0.001904$$

$$F_D = C_D \frac{1}{2} \rho A U^2 = (0.001904) \frac{1}{2} (1000)(40 \times 8)(10)^2 = 30{,}464 \ N$$

To solve this problem by using power law, we need to determine the velocity profile. Reynolds number is :

$$Re = \frac{Ux}{\nu} = \frac{(10)(40)}{1.519 \times 10^{-6}} = 263{,}331{,}138 \approx 2.6 \times 10^{8}$$

From previous lesson, we could determine the velocity profile as:

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{9}}$$

$n = 7 \quad for \quad 5 \times 10^{5} < Re < 10^{7}$
$n = 8 \quad for \quad 10^{7} < Re < 10^{8}$
$n = 9 \quad for \quad 10^{8} < Re < 10^{9}$

From textbook, for $n = 9$, shear stress from Blasius calculation can be determined as:

$$\tau = 0.023\rho U^2 \left(\frac{\nu}{U\delta}\right)^{\frac{1}{4}} = 0.023\rho U^2 \left(\frac{\mu}{\rho U\delta}\right)^{\frac{1}{4}}$$

Shear stress from Von-Karman equation can be determined:

$$\tau = \rho U^2 \frac{d\theta}{dx} = \frac{9}{110}\rho U^2 \frac{d\delta}{dx}$$

Equating these two shear stresses.

We could find the boundary layer thickness is:

$$\delta = \frac{0.433x}{(Re)^{\frac{1}{5}}} = 0.358 \ m$$

δ from empirical method is 0.384 m

δ from power-law method is 0.358 m

Result from power-law method is lower.
The difference is about 6.77%

Drag force:

$$F_D = C_D \frac{1}{2} \rho A U^2 = \frac{0.071}{(Re)^{\frac{1}{5}}} \times \frac{1}{2} (1000)(40 \times 8)(10^2) = 23{,}511 \ N$$

F_D from empirical method is 30,464 N

F_D from power-law method is 23,511 N

Result from power-law method is lower.
The difference is about 22.82%