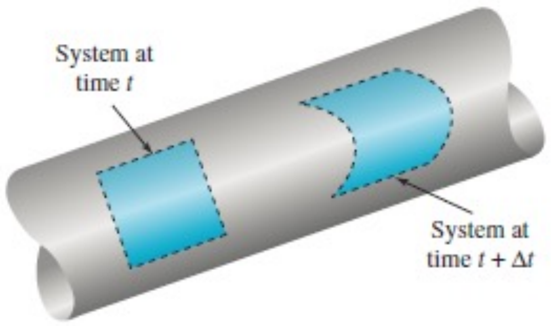
The Integral Forms of the Fundamental Laws

The Three Basic Laws

- The integral quantities in fluid mechanics are contained in the three laws:
  - Conservation of Mass
  - First Law of Thermodynamics
  - Newton's Second Law

- They are expressed using a Lagrangian description in terms of a system (fixed collection of material particles).



Example of a system in fluid mechanics.

The Three Basic Laws

- **CONSERVATION OF MASS: Mass of a system remains constant.**

$$\frac{D}{Dt}\int_{sys} \rho \, d\forall = 0$$

Integral form of the mass-conservation equation. ρ = Density; d$\forall$ = Volume occupied by the particle

- **FIRST LAW OF THERMODYNAMICS: Rate of heat transfer to a system minus the rate at which the system does work equals the rate at which the energy of the system is changing.**

$$\dot{Q} - \dot{W} = \frac{D}{Dt}\int_{sys} e\rho \, d\forall$$

Specific energy (e): Accounts for kinetic energy per unit mass ($0.5V^2$), potential energy per unit mass (gz), and internal energy per unit mass ($\tilde{u}$).

The Three Basic Laws

- **NEWTON'S SECOND LAW: Resultant force acting on a system equals the rate at which the momentum of the system is changing.**

$$\Sigma \mathbf{F} = \frac{D}{Dt}\int_{\text{sys}} \mathbf{V}\rho\, d\mathcal{V}$$

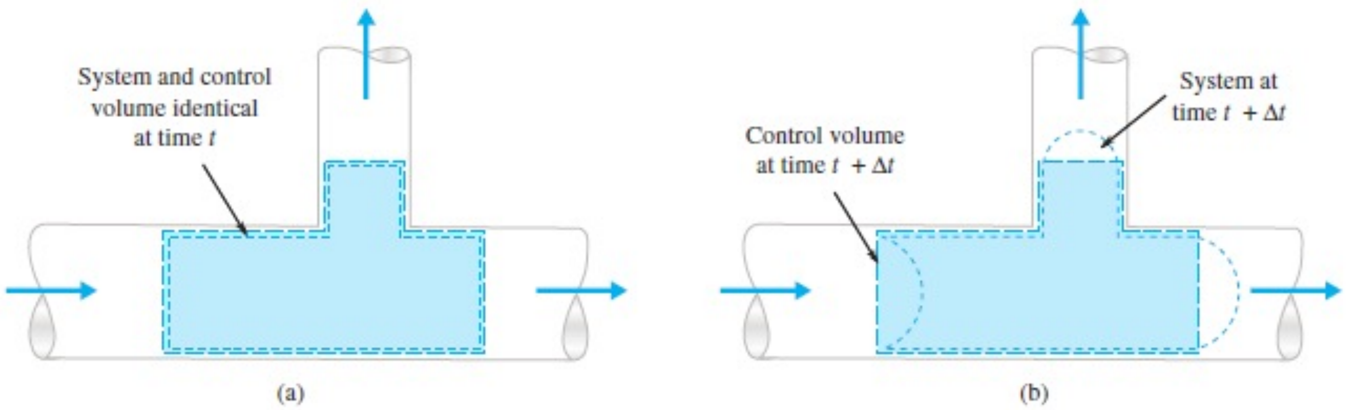
In an inertial frame of reference.

- **Moment-of-Momentum Equation: Resultant moment acting on a system equals the rate of change of the angular momentum of the system.**

$$\Sigma \mathbf{M} = \frac{D}{Dt}\int_{\text{sys}} \mathbf{r} \times \mathbf{V}\rho\, d\mathcal{V}$$

The Three Basic Laws

- **Control Volume**: A region of space into which fluid enters and/or from which fluid leaves.



Example of a fixed control volume and a system: (a) time t; (b) time $t + \Delta t$.

System-to-Control-Volume Transformation

- Interested in the time rate of change of an extensive property to be expressed in terms of quantities related to a control volume.
  - Involves fluxes of an extensive property in and out of a control volume.
  - **Flux** is the measure of the rate at which an extensive property crosses an area.

System-to-Control-Volume Transformation

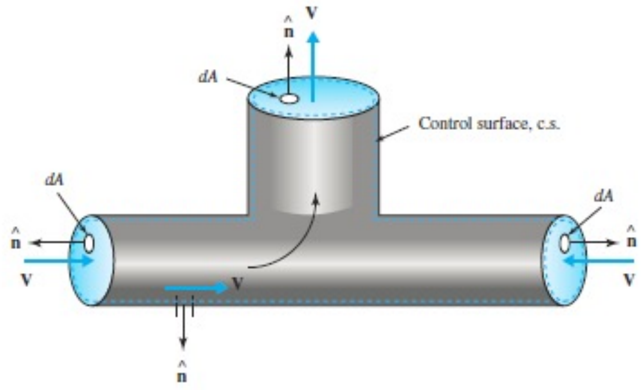


Illustration showing the flux of an extensive property.

Control surface: The surface area that completely encloses the control volume.

- The flux across an element dA is:

$$\text{flux across } dA = \eta \rho \hat{\mathbf{n}} \cdot \mathbf{V}\, dA$$

$\hat{n}$: Unit vector normal to dA (<u>always points out</u> of the control volume)
η: Intensive property

- Only the normal component of $\hat{n}$·V contributes to this flux.
  - Positive component means a flux out of the volume.
  - Negative component indicates a flux into the volume.
  - If the net flux is positive: Flux out > flux in

System-to-Control-Volume Transformation

Reynolds Transport Theorem

- The Reynolds transport theorem is a system-to-control-volume transformation.

$$\frac{DN_{sys}}{Dt} = \frac{d}{dt}\int_{c.v.} \eta\rho \, dV + \int_{c.s.} \eta\rho\hat{\mathbf{n}}\cdot\mathbf{V} \, dA$$

- This is a Lagrangian-to-Eulerian transformation of the rate of change of an extensive quantity.
  - First part of integral: Rate of change of an extensive property in the control volume.
  - Second part of integral: Flux of the extensive property across the control surface (nonzero where fluid crosses the control surface).

System-to-Control-Volume Transformation

Reynolds Transport Theorem

$$\frac{DN_{sys}}{Dt} = \frac{d}{dt}\int_{c.v.} \eta\rho \; d\mathcal{V} + \int_{c.s.} \eta\rho \hat{\mathbf{n}} \cdot \mathbf{V} \, dA$$

- An equivalent form of the control volume is:

$$\frac{DN_{sys}}{Dt} = \int_{c.v.} \frac{\partial}{\partial t}(\rho\eta) \, d\mathcal{V} + \int_{c.s.} \eta\rho \hat{\mathbf{n}} \cdot \mathbf{V} \, dA$$

- The time derivative of the control volume is moved inside the integral:
  - For a fixed control volume, the limits on the volume integral are independent of time.

System-to-Control-Volume Transformation

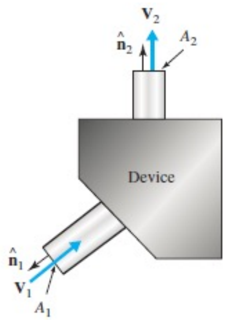
- **4.3.1 Simplifications of the Reynolds Transport Theorem**

$$\frac{DN_{sys}}{Dt} = \frac{d}{dt}\int_{c.v.} \eta\rho \, d\mathcal{V} + \int_{c.s.} \eta\rho \hat{\mathbf{n}} \cdot \mathbf{V} \, dA$$

$$\frac{DN_{sys}}{Dt} = \int_{c.v.} \frac{\partial}{\partial t}(\rho\eta) \, d\mathcal{V} + \int_{c.s.} \eta\rho \hat{\mathbf{n}} \cdot \mathbf{V} \, dA$$

- **Steady-state (time derivative is zero)**: $$\frac{DN_{sys}}{Dt} = \int_{c.v.} \eta\rho \, \hat{\mathbf{n}} \cdot \mathbf{V} \, dA$$

- **Often one inlet (A_1), and one outlet (A_2)**: $$\frac{DN_{sys}}{Dt} = \int_{A_2} \eta_2\rho_2 V_2 \, dA - \int_{A_1} \eta_1\rho_1 V_1 \, dA$$



- **For uniform properties over a plane area**: $$\frac{DN_{sys}}{Dt} = \eta_2\rho_2 V_2 A_2 - \eta_1\rho_1 V_1 A_1$$

System-to-Control-Volume Transformation

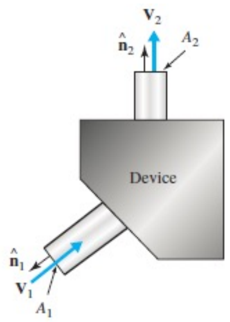
- **4.3.1 Simplifications of the Reynolds Transport Theorem**

$$\frac{DN_{sys}}{Dt} = \frac{d}{dt}\int_{c.v.} \eta\rho \, dV + \int_{c.s.} \eta\rho\hat{\mathbf{n}}\cdot\mathbf{V}\, dA$$

$$\frac{DN_{sys}}{Dt} = \int_{c.v.} \frac{\partial}{\partial t}(\rho\eta)\, dV + \int_{c.s.} \eta\rho\hat{\mathbf{n}}\cdot\mathbf{V}\, dA$$

- **Unsteady flow with uniform flow properties**:

$$\frac{DN_{sys}}{Dt} = V_{c.v.}\frac{d(\eta\rho)}{dt} + \eta_2\rho_2 V_2 A_2 - \eta_1\rho_1 V_1 A_1$$



Conservation of Mass

$$\frac{Dm_{sys}}{Dt} = \frac{D}{Dt}\int_{sys} \rho \; d\forall = 0$$

Mass of a system is fixed.

- For a steady flow, this simplifies to:

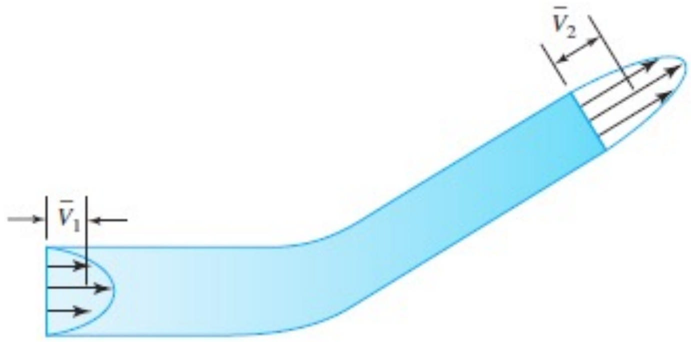
$$\int_{c.s.} \rho \hat{\mathbf{n}} \cdot \mathbf{V} \, dA = 0$$

- Uniform flow with one entrance and one exit:

$$\boxed{\rho_2 A_2 V_2 = \rho_1 A_1 V_1}$$

For constant density, the continuity equation is only dependent on A and V

Conservation of Mass



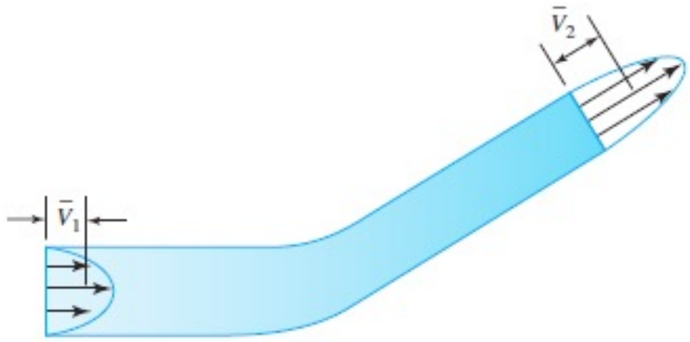
Nonuniform velocity profiles.

- If the density is uniform over each area, with nonuniform velocity profiles:

$$\rho_1 \int_{A_1} V_1\, dA = \rho_2 \int_{A_2} V_2\, dA \qquad \rho_1 \overline{V}_1 A_1 = \rho_2 \overline{V}_2 A_2$$ (averages can also be used)

- The *mass flux* $\dot{m}$ (kg/s) is the mass rate of flow: $\dot{m} = \int_A \rho V_n\, dA$
  - Where V_n is the normal component of velocity.

Conservation of Mass



Nonuniform velocity profiles.

- The *flow rate (*or *discharge)* Q (m^3/s) is the volume rate of flow: $Q = \int_A V_n \, dA$
    - Mass flow rate is often used in compressible flow. The flow rate is often used to specify incompressible flow.

Conservation of Mass

Water flows at a uniform velocity of 3 m/s into a nozzle that reduces the diameter from 100 mm to 20 mm (Figure E4.1). Calculate the water's velocity leaving the nozzle and the flow rate.

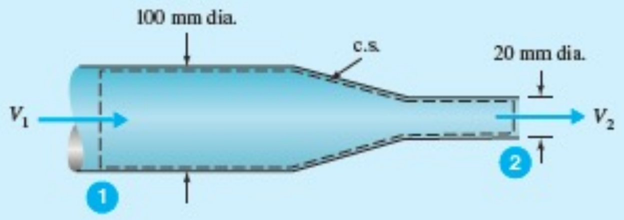


Figure E4.1

Solution

The control volume is selected to be the inside of the nozzle as shown. Flow enters the control volume at section 1 and leaves at section 2. The simplified continuity equation (4.4.6) is used since the density of water is assumed constant and the velocity profiles are uniform:

$$A_1V_1 = A_2V_2$$

$$\therefore V_2 = V_1\frac{A_1}{A_2} = 3\,\frac{\pi \times 0.1^2/4}{\pi \times 0.02^2/4} = \underline{75 \text{ m/s}}$$

The flow rate, or discharge, is found to be

$$Q = V_1A_1$$
$$= 3 \times \pi \times 0.1^2/4 = \underline{0.0236 \text{ m}^3\text{/s}}$$

Conservation of Mass

Water flows in and out of a device as shown in Figure E4.2a. Calculate the rate of change of the mass of water (dm/dt) in the device.

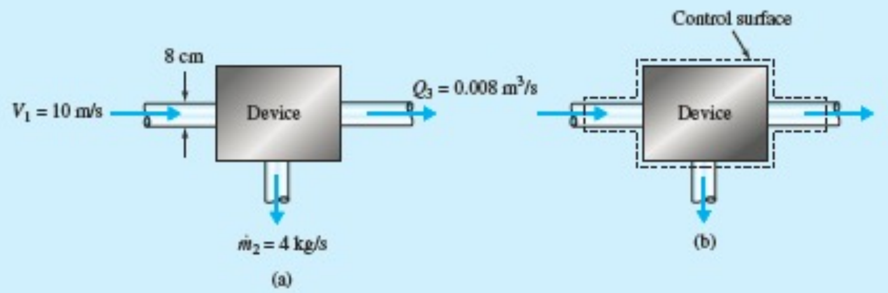


Figure E4.2

Solution

The control surface of the control volume selected is shown in Figure E4.2b. The continuity equation (4.4.2), with three surfaces across which water flows, takes the following form:

$$0 = \frac{d}{dt}\int_{\text{c.v.}} \rho \, d\mathcal{V} + \int_{\text{c.s.}} \rho \hat{\mathbf{n}} \cdot \mathbf{V} \, dA$$

$$= \frac{dm}{dt} - \rho_1 A_1 V_1 + \rho_2 A_2 V_2 + \rho_3 A_3 V_3$$

where we have assumed the density to be constant over the volume and we have used $\mathbf{V}_1 \cdot \hat{\mathbf{n}} = -V_1$, since $\hat{\mathbf{n}}_1$ points out of the volume, opposite to the direction of $\mathbf{V}_1$. The last three terms come from the area integral. In terms of the quantities given, the above can be expressed as

$$0 = \frac{dm}{dt} - \rho_1 A_1 V_1 + \dot{m}_2 + \rho_3 Q_3$$

$$= \frac{dm}{dt} - 1000 \text{ kg/m}^3 \times \left(\pi \times \frac{0.04^2}{10\,000}\right) \text{m}^2 \times 10 \text{ m/s} + 4 \text{ kg/s}$$

$$+ 1000 \text{ kg/m}^3 \times (0.008) \text{ m}^3/\text{s}$$

This is solved to yield

$$\frac{dm}{dt} = \underline{\underline{38.3 \text{ kg/s}}}$$

Hence the mass is increasing at the rate of 38.3 kg/s. To accomplish this, the device could contain a spongelike material that absorbs water.

Conservation of Mass

A uniform flow of air approaches a cylinder as shown in Figure E4.3a. The symmetrical velocity distribution at the location shown downstream in the wake of the cylinder is approximated by

$$u(y) = 1.25 + \frac{y^2}{4} \qquad -1 < y < 1$$

where $u(y)$ is in m/s and y is in meters. Determine the mass flux across the surface AB per meter of depth (into the paper). Use $\rho = 1.23 \text{ kg/m}^3$.

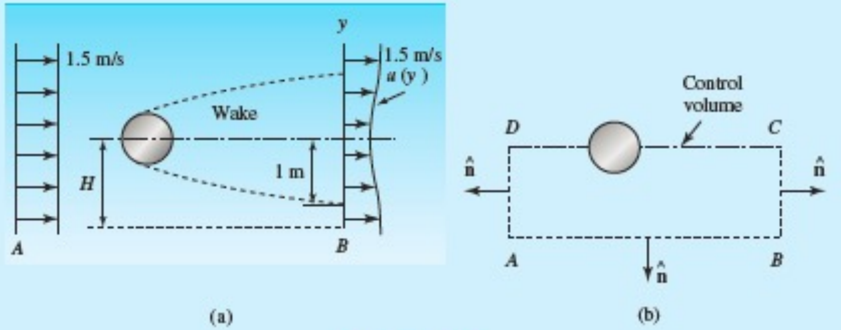


Figure E4.3

Solution

Select $ABCD$ as the control volume (Figure E4.3b). Outside the wake (a region of retarded flow) the velocity is constant at 1.5 m/s. Hence the velocity normal to plane AD is 1.5 m/s. No mass flux crosses the surface CD because of symmetry. Assuming a steady flow, the continuity equation (4.4.3) becomes

$$0 = \int_{\text{c.s.}} \rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA$$

Mass flux occurs across three surfaces: AB, BC, and AD. Thus the equation above takes the form

$$\begin{aligned} 0 &= \int_{A_{AB}} \rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA + \int_{A_{BC}} \rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA + \int_{A_{AD}} \rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA \\ &= \dot{m}_{AB} + \int_0^H \rho u(y)\, 1 \times dy - \rho \text{ kg/m}^3 \times 1.5 \text{ m/s} \times H \text{ m} \times 1 \text{ m} \end{aligned}$$

where the negative sign for surface AD results from the fact that the unit vector points out of the volume to the left while the velocity vector points to the right. Recall that a negative sign in the steady-flow continuity equation is always associated with an influx and a positive sign with an outflux. Now, we integrate out to 1 m instead of H, since the mass that enters on the left beyond 1 m simply leaves on the right with no net gain or loss. So, letting $H = 1$ m, we have

$$0 = \dot{m}_{AB} + \int_0^1 1.23\left(1.25 + \frac{y^2}{4}\right) dy - (1.23 \times 1 \times 1.5)$$

Perform the integration and there results

$$\dot{m}_{AB} = \underline{0.205 \text{ kg/s per meter}}$$

Conservation of Mass

A balloon is being inflated with a water supply of 0.6 m³/s (Figure E4.4a). Find the rate of growth of the radius at the instant when $R = 0.5$ m.

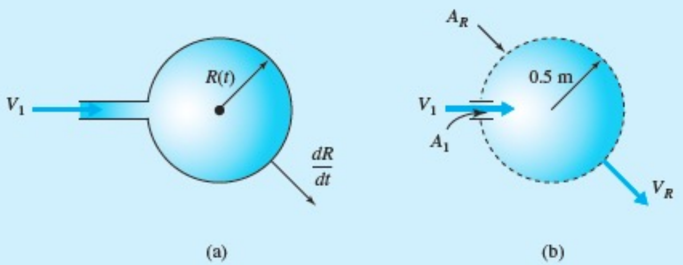


Figure E4.4

Solution

The objective is to find dR/dt when the radius $R = 0.5$ m. This growth rate $V_R = dR/dt$ is the same as the water velocity normal to the wall of the balloon. Therefore, we select as our fixed control volume a sphere with a constant radius of 0.5 m (see Figure E4.4b) so that we can calculate the velocity of the water at the surface at the instant shown moving radially out at $R = 0.5$ m. The continuity equation is written as

$$0 = \int_{c.v.} \underbrace{\frac{\partial \rho}{\partial t}}_{0} d\mathcal{V} + \int_{c.s.} \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA$$

The first term is zero because the density of water inside the control volume does not change in time. Further, the water crosses two areas: the inlet area A_1 with a velocity V_1 and the remainder of the sphere surface A_R with a velocity V_R. We will assume that $A_1 << A_R$. The continuity equation then takes the form

$$0 = -\rho A_1 V_1 + \rho A_R V_R$$

Since the flow rate into the volume is $A_1 V_1 = 0.6\ \text{m}^3/\text{s}$ and $A_R \simeq 4\pi R^2$ assuming that A_1 is quite small, we can solve for V_R. At $R = 0.5$ m

$$V_R = \frac{A_1 V_1}{4\pi R^2} = \frac{0.6\ \text{m}^3/\text{s}}{4\pi \times 0.5^2\ \text{m}^2} = 0.191\ \text{m/s}$$

$$\therefore \frac{dR}{dt} = \underline{\underline{0.191\ \text{m/s}}}$$

We have used a fixed control volume and allowed the moving surface of the balloon to pass through it at the instant considered. With this approach it is possible to model situations in which surfaces, such as a piston, are allowed to move.

Conservation of Mass

This example shows that there may be more than one good choice for a control volume. We want to determine the rate at which the water level rises in an open container if the water coming in through a 0.10-m² pipe has a velocity of 0.5 m/s and the flow rate going out is 0.2 m³/s (Figure E4.5a). The container has a circular cross section with a diameter of 0.5 m.

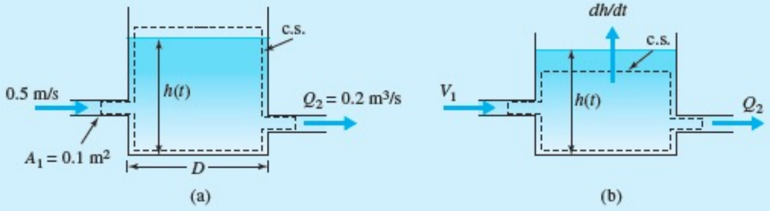


Figure E4.5

Solution

First we select a control volume that extends above the water surface as shown in Figure E4.5a. Apply the continuity equation (Eq. 4.4.2):

$$\frac{d}{dt}\int_{\text{c.v.}} \rho\, d\mathcal{V} + \rho(-V_1)A_1 + \rho V_2 A_2 = 0$$

in which the first term describes the rate of change of mass in the control volume. Hence, neglecting the airmass above the water, we have

$$\frac{d(\rho h \pi D^2/4)}{dt} - \rho V_1 A_1 + \rho Q_2 = 0$$

Divide by the constant ρ,

$$\frac{\pi D^2}{4}\frac{dh}{dt} - V_1 A_1 + Q_2 = 0$$

The rate at which the water level is rising is then

$$\frac{dh}{dt} = \frac{V_1 A_1 - Q_2}{\pi D^2/4}$$

Thus

$$\frac{dh}{dt} = \frac{(0.5 \times 0.1 - 0.2)\ \text{m}^3/\text{s}}{(\pi \times 0.5^2/4)\ \text{m}^2} = \underline{-0.764\ \text{m/s}}$$

The negative sign indicates that the water level is actually decreasing.

Let's solve this problem again but with another choice for the control volume, one with its top surface below the water level (Figure E4.5b). The velocity at the top surface is then equal to the rate at which the surface rises, i.e., dh/dt. The flow condition inside the control volume is steady. Hence we can apply Eq. 4.4.4. There are three areas across which fluid flows. On the third area, the velocity is dh/dt; hence the continuity equation takes the form

$$\rho(-V_1)A_1 + \rho Q_2 + \rho\frac{dh}{dt}\frac{\pi}{4}D^2 = 0$$

so that

$$\frac{dh}{dt} = \frac{V_1 A_1 - Q_2}{\pi D^2/4}$$

This is the same result as given above.

Energy Equation

- This equation is required if heat is transferred (boiler/compressor) or work is done (pump/turbine).
  - Can relate pressures/velocities when Bernoulli's equation cannot be used.

$$\dot{Q} - \dot{W} = \frac{D}{Dt} \int_{\text{sys}} e\rho \, d\forall$$

Where e is the specific energy and consists of the specific kinetic energy, specific potential energy, and specific internal energy.

$$e = \frac{V^2}{2} + gz + \overline{u}$$

- In terms of a control volume: $\dot{Q} - \dot{W} = \dfrac{d}{dt} \displaystyle\int_{\text{c.v.}} e\rho \, d\forall + \int_{\text{c.s.}} \rho e \mathbf{V} \cdot \hat{\mathbf{n}} \, dA$

Energy Equation

- In terms of a control volume: $\dot{Q} - \dot{W} = \frac{d}{dt}\int_{\text{c.v.}} e\rho \, d\mathcal{V} + \int_{\text{c.s.}} \rho e \mathbf{V} \cdot \hat{\mathbf{n}} \, dA$
- $\dot{Q}$: Rate-of-energy transfer across the control surface due to a temperature difference.
- $\dot{W}$: Work-rate term due to work being done by the system.

Energy Equation

- **Work-Rate Term**

- The work-rate term is from the work being done by the system.
- Rate of work (Power) is the dot product of force with its velocity.

$$\dot{W} = -\mathbf{F} \cdot \mathbf{V}_I$$

The velocity is measured w.r.t. a fixed inertial reference frame. Negative sign is because work done on the control volume is negative.

- If the force is from variable stress over a control surface:

$$\dot{W} = -\int_{\text{c.s.}} \boldsymbol{\tau} \cdot \mathbf{V}_I \, dA$$

  - τ is a stress vector acting on an elemental area dA [A differential force].

Energy Equation

- **Work-Rate Term**

$$\dot{W} = \int_{\text{c.s.}} p\hat{\mathbf{n}} \cdot \mathbf{V}\, dA + \dot{W}_S + \dot{W}_{\text{shear}} + \dot{W}_I$$

$\int p\hat{\mathbf{n}} \cdot \mathbf{V}\, dA$ Work rate resulting from the force due to pressure moving at the control surface. It is often referred to as **flow work**.

$\dot{W}_S$ Work rate resulting from rotating shafts such as that of a pump or turbine, or the equivalent electric power.

$\dot{W}_{\text{shear}}$ Work rate due to the shear acting on a moving boundary, such as a moving belt.

$\dot{W}_I$ Work rate that occurs when the control volume moves relative to a fixed reference frame.

Energy Equation

- **General Energy Equation**

- From the previous equation, the energy equation can be rewritten as:

$$\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_I = \frac{d}{dt}\int_{\text{c.v.}} e\rho \, d\forall + \int_{\text{c.s.}} \left(e + \frac{p}{\rho}\right)\rho \hat{\mathbf{n}} \cdot \mathbf{V} \, dA$$

- Losses are the sum of all terms for unusable forms of energy.

$$\text{losses} = -\dot{Q} + \frac{d}{dt}\int_{\text{c.v.}} \tilde{u} \, \rho \, dV + \int_{\text{c.s.}} \tilde{u}\rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA$$

  - Can be due to viscosity (causes friction resulting in increased internal energy).
  - Or due to changes in geometry resulting in separated flows.

Energy Equation

- **Steady Uniform Flow**

- For steady-flow with one inlet and one outlet (with uniform profile) and no shear work, the following energy equation is used:

$$-\frac{\dot{W}_S}{\dot{m}g} = \frac{V_2^2 - V_1^2}{2g} + \frac{p_2}{\gamma_2} - \frac{p_1}{\gamma_1} + z_2 - z_1 + h_L$$

- Where h_L is the head loss (dimensions of length).

$$h_L = \frac{\tilde{u}_2 - \tilde{u}_1}{g} - \frac{\dot{Q}}{\dot{m}g}$$

$$h_L = K\frac{V^2}{2g}$$

Where **K** is the loss coefficient

- $\frac{V^2}{g}$ is the velocity head, and $\frac{p}{\gamma}$ is the pressure head.

Energy Equation

- **Steady Uniform Flow**

- For steady-flow with one inlet and one outlet (with uniform profiles) and no shear work, negligible losses, and no shaft work:

$$\frac{V_2^2}{2g} + \frac{p_2}{\gamma_2} + z_2 = \frac{V_1^2}{2g} + \frac{p_1}{\gamma_1} + z_1$$

Almost identical to Bernoulli's equation for a constant density flow.

- The pump head, H_P is the energy term associated for a pump $[\frac{\dot{W}_S}{\dot{m}g}]$. If a turbine is involved, the energy term is called the turbine head.

Energy Equation

- **Steady Uniform Flow**

- If a turbine/pump is used, the efficiency of a device is needed, η_T
  - The power generated by the turbine is:

$$\dot{W}_T = \dot{m} g H_T \eta_T = \gamma Q H_T \eta_T$$

$$\dot{m} = \rho A V$$

  - The power required by a pump is:

The power is calculated in Watts

$$\dot{W}_P = \frac{\dot{m} g H_P}{\eta_P} = \frac{\gamma Q H_P}{\eta_P}$$

Energy Equation

- **Steady Nonuniform Flow**

- If a uniform velocity profile assumption cannot be used, the velocity distribution should be corrected:
  - Using a **kinetic-energy correction factor α**

$$\alpha = \frac{\int V^3 dA}{\bar{V}^3 A}$$

  - The term that accounts for the flux in kinetic energy is:

$$\frac{1}{2}\rho\int_A V^3\, dA = \frac{1}{2}\alpha\rho\bar{V}^3 A$$

With $\bar{V}$ being the average velocity over area A

- The final equation that account for this nonuniform velocity distribution is:

$$H_P + \alpha_1\frac{\bar{V}_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = H_T + \alpha_2\frac{\bar{V}_2^2}{2g} + \frac{p_2}{\gamma} + z_2 + h_L$$

Energy Equation

The pump of Figure E4.6 is to increase the pressure of 0.2 m^3/s of water from 200 kPa to 600 kPa. If the pump is 85% efficient, how much electrical power will the pump require? The exit area is 20 cm above the inlet area. Assume inlet and exit areas are equal.

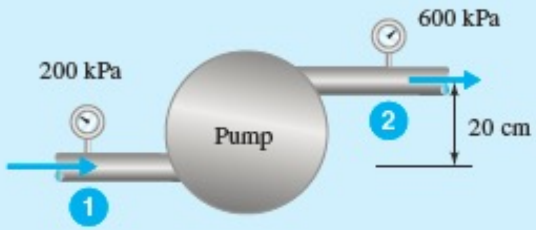


Figure E4.6

Solution

The energy equation (4.5.24) across the pump provides the energy delivered to the water as a pump head:

$$H_P = \frac{p_2 - p_1}{\gamma} + z_2 - z_1$$

$$= \frac{(600\,000 - 200\,000)\,\mathrm{N/m^2}}{9810\,\mathrm{N/m^3}} + 0.2\,\mathrm{m} = 41.0\,\mathrm{m}$$

where $V_2 = V_1$ since the inlet and exit areas are equal, and any losses are accounted for with the efficiency of Eq. 4.5.26. That equation provides the power required by the pump:

$$\dot{W}_P = \frac{\gamma Q H_P}{\eta_P}$$

$$= \frac{9810\,\mathrm{N/m^3} \times 0.2\,\mathrm{m^3/s} \times 41.0\,\mathrm{m}}{0.85} = 94\,600\,\mathrm{J/s} \quad \text{or} \quad \underline{94.6\,\mathrm{kW}}$$

Energy Equation

Water flows from a reservoir through a 800-mm-diameter pipeline to a turbine-generator unit and exits to a river that is 30 m below the reservoir surface. If the flow rate is 3 m^3/s, and the turbine-generator efficiency is 88%, calculate the power output. Assume the loss coefficient in the pipeline (including the exit) to be $K = 2$.

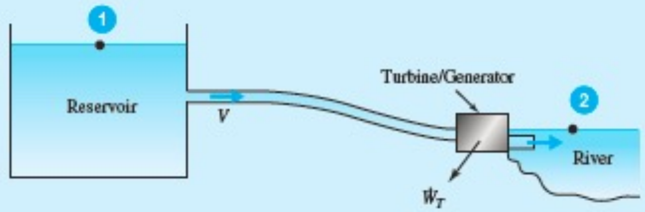


Figure E4.7

Solution

Referring to Figure E4.7, we select the control volume to extend from section 1 to section 2 on the reservoir and river surfaces, where we know the velocities, pressures, and elevations; we consider the water surface of the left reservoir to be the entrance and the water surface of the river to be the exit. The velocity in the pipe is

$$V = \frac{Q}{A} = \frac{3}{\pi \times 0.8^2/4} = 5.97 \text{ m/s}$$

Now, consider the energy equation. We will use gage pressures so that $p_1 = p_2 = 0$; the datum is placed through the lower section 2 so that $z_2 = 0$; the velocities V_1 and V_2 on the reservoir surfaces are negligibly small; K is assumed to be based on the 800 mm-diameter pipe velocity. The energy equation (4.5.24) then becomes

$$\cancelto{0}{H_P} + \cancelto{0}{\frac{V_1^2}{2g}} + \cancelto{0}{\frac{p_1}{\gamma}} + z_1 = H_T + \cancelto{0}{\frac{V_2^2}{2g}} + \cancelto{0}{\frac{p_2}{\gamma}} + \cancelto{0}{z_2} + K\frac{V^2}{2g}$$

$$30 = H_T + 2\frac{5.97^2 \text{ m}^2/\text{s}^2}{2 \times 9.81 \text{ m/s}^2}$$

$$\therefore H_T = 26.4 \text{ m}$$

From this the power output is found using Eq. 4.5.25 to be

$$\dot{W}_T = 3 \text{ m/s}^3 \times 9810 \text{ N/m}^3 \times 26.4 \text{ m} \times 0.88 = \underline{684 \text{ kW}}$$

In this example we have used gage pressure; the potential-energy datum was assumed to be placed through section 2, V_1 and V_2 were assumed to be insignificantly small, and K was assumed to be based on the 762-mm-diameter pipe velocity.

Energy Equation

The venturi meter shown reduces the pipe diameter from 100 mm to a minimum of 50 mm (Figure E4.8). Calculate the flow rate and the mass flux assuming ideal conditions.

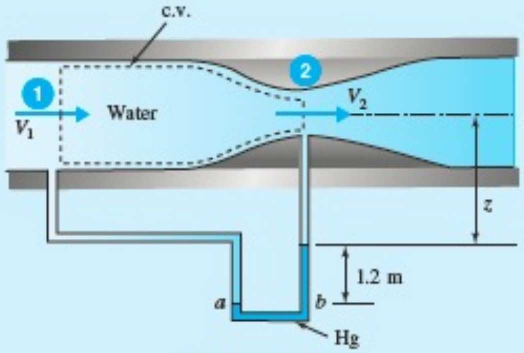


Figure E4.8

Solution

The control volume is selected as shown such that the entrance and exit correspond to the sections where the pressure information of the manometer can be applied. The manometer's reading is interpreted as follows:

$$p_a = p_b$$
$$p_1 + \gamma(z + 1.2) = p_2 + \gamma z + 13.6\gamma \times 1.2$$

where z is the distance from the pipe centerline to the top of the mercury column. The manometer then gives

$$\frac{p_1 - p_2}{\gamma} = (13.6 - 1) \times 1.2 = 15.12\,\text{m}$$

Energy Equation

Continuity (4.4.6) allows us to relate V_2 to V_1 by

$$V_1A_1 = V_2A_2$$

$$\therefore\ V_2 = \frac{A_1}{A_2}V_1 = \frac{\pi \times 10^3/4}{\pi \times 5^2/4}V_1 = 4V_1$$

The energy equation (4.5.17) assuming ideal conditions (no losses and uniform flow) with $h_L = \dot{W}_S = 0$ takes the form

$$0 = \frac{V_2^2 - V_1^2}{2g} + \frac{p_2 - p_1}{\gamma} + \cancelto{0}{(z_2 - z_1)}$$

$$= \frac{16V_1^2 - V_1^2}{2g} - 15.12$$

$$\therefore\ V_1 = 4.45\text{m/s}$$

The flow rate is

$$Q = A_1V_1 = (\pi \times 0.05^2) \times 4.45 = \underline{0.0350\ \text{m}^3/\text{s}}$$

The mass flux is

$$\dot{m} = \rho Q = 1000 \times 0.035 = \underline{35.0\ \text{kg/s}}$$

Energy Equation

The velocity distribution for a certain flow in a pipe is $V(r) = V_{max}(1 - r^2/r_0^2)$, where r_0 is the pipe radius (Figure E4.9). Determine the kinetic-energy correction factor.

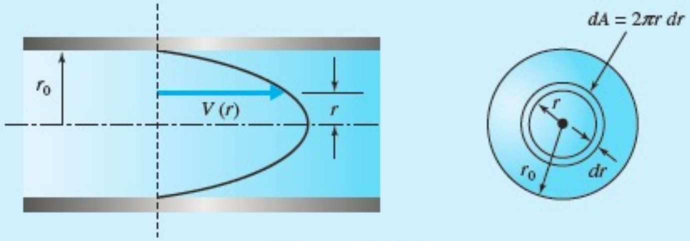


Figure E4.9

Solution

To find the kinetic-energy correction factor α, we must know the average velocity. It is (combine Eqs. 4.4.10 and 4.4.11)

$$\bar{V} = \frac{\int V dA}{A}$$

$$= \frac{1}{\pi r_0^2}\int_0^{r_0} V_{max}\left(1 - \frac{r^2}{r_0^2}\right)2\pi r\,dr \quad = \frac{2\pi V_{max}}{\pi r_0^2}\int_0^{r_0}\left(r - \frac{r^3}{r_0^2}\right)dr$$

$$= \frac{2V_{max}}{r_0^2}\left(\frac{r_0^2}{2} - \frac{r_0^4}{4r_0^2}\right) = \frac{1}{2}V_{max}$$

Using Eq. 4.5.27, there results

$$\alpha = \frac{\int V^3 dA}{\bar{V}^3 A}$$

$$= \frac{\int_0^{r_0} V_{max}^3\,(1 - r^2/r_0^2)^3 2\pi r\,dr}{\left(\frac{1}{2}V_{max}\right)^3 \pi r_0^2} \quad = \frac{16}{r_0^2}\int_0^{r_0}\left(1 - \frac{3r^2}{r_0^2} + \frac{3r^4}{r_0^4} - \frac{r^6}{r_0^6}\right)r\,dr$$

$$= \frac{16}{r_0^2}\left(\frac{r_0^2}{2} - \frac{3r_0^2}{4} + \frac{3r_0^2}{6} - \frac{r_0^2}{8}\right) = 2$$

Consequently, the kinetic energy flux associated with a parabolic velocity distribution across a circular area is given by

$$\int \rho \mathbf{V}\cdot\hat{\mathbf{n}}\frac{V^2}{2}dA = 2 \times \frac{\dot{m}\bar{V}^2}{2} = \dot{m}\bar{V}^2$$

Parabolic velocity distributions are encountered in laminar flows in pipes and between parallel plates, downstream of inlets and geometry changes (valves, elbows, etc.). The Reynolds number must be quite small, usually less than about 2000.

Energy Equation

The drag force on an automobile (Figure E4.10) is approximated by the expression $0.15\rho V_\infty^2 A$, where A is the projected cross-sectional area and V_∞ is the automobile's speed. If $A = 1.2 \text{ m}^2$, calculate the efficiency η of the engine if the rate of fuel consumption $\dot{f}$ (the gas mileage) is 15×10^3 km/m³ and the automobile travels at 90 km/h. Assume that the fuel releases 44 000 kJ/kg during combustion. Neglect the energy lost due to the exhaust gases and coolant and assume that the only resistance to motion is the drag force. Use $\rho_{\text{air}} = 1.12 \text{ kg/m}^3$ and $\rho_{\text{fuel}} = 680 \text{ kg/m}^3$.

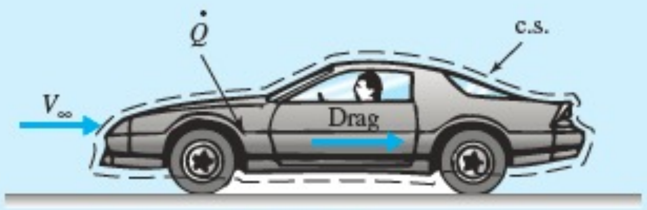


Figure E4.10

Solution

If the car is taken as the moving control volume (note that the control volume is fixed), as shown, we can simplify the energy equation (Eq. 4.5.3 in combination with 4.5.11) to

$$\dot{Q} - \dot{W}_I = 0$$

Energy Equation

since all other terms are negligible; there is no velocity crossing the control volume, so $\mathbf{V} \cdot \hat{\mathbf{n}} = 0$ (neglect the energy of the exhaust gases); there is no shear or shaft work; the energy of the c.v. remains constant. The energy input $\dot{Q}$ which accomplishes useful work is η times the energy released during combustion; that is,

$$\dot{Q} = \dot{m}_f \times 44\,000\eta \text{ kJ/s}$$

where $\dot{m}_f$ is the mass flux of the fuel. The mass flux of fuel is determined knowing the rate of fuel consumption $\dot{f}$ and the density of fuel as 680 kg/m³, as follows:

$$\dot{f} = \frac{\text{distance}}{\text{volume}} = \frac{V_\infty \times \text{time}}{Q \times \text{time}} = \frac{V_\infty}{\dot{m}_f/\rho_f} = \frac{\rho_f V_\infty}{\dot{m}_f}$$

with $V_\infty = 90\,000/3600 = 25$ m/s, we have, using $\dot{f} = 15 \times 10^6$ m/m³,

$$15 \times 10^6 = \frac{680 \times 25}{\dot{m}_f}$$

$$\therefore\ \dot{m}_f = 0.001133 \text{ kg/s}$$

The inertial work-rate term is

$$\begin{aligned}\dot{W}_I &= V_\infty \times \text{drag}\\ &= 0.15\rho V_\infty^3 A = 0.15 \times 1.12 \times 25^3 \times 1.2 = 3150 \text{ J/s}\end{aligned}$$

Equating $\dot{Q} = \dot{W}_I$, we have

$$44\,000\eta \times 0.001133 = 3.15$$

$$\therefore\ \eta = 0.0632 \quad \text{or} \quad \underline{6.32\%}$$

This is obviously a very low percentage, perhaps surprisingly low to the reader. Very little power (3.15 kJ/s = 4.22 hp) is actually needed to propel the automobile at 90 km/h. The relatively large engine, needed primarily for acceleration, is quite inefficient when simply propelling the automobile.

Note the importance of using a stationary reference frame. The reference frame attached to the automobile is an inertial reference frame since it is moving at constant velocity. Yet the energy equation demands a stationary reference frame allowing the energy required by the drag force to be properly included.

Momentum Equation

- **General Momentum Equation**

- Newton's second law (momentum equation): The resultant force acting on a system equals the rate of change of momentum of the system in an inertial reference frame.

$$\Sigma \mathbf{F} = \frac{D}{Dt}\int_{\text{sys}} \rho \mathbf{V}\, d\forall$$

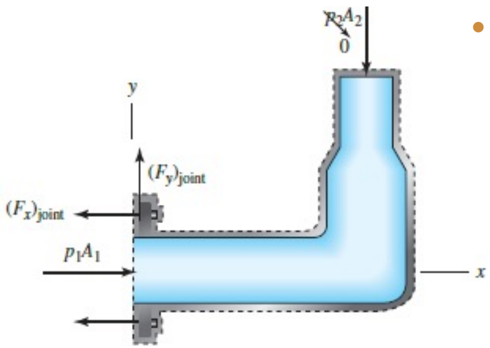
- For a control volume: $\Sigma \mathbf{F} = \frac{d}{dt}\int_{\text{c.v.}} \rho \mathbf{V}\, d\forall + \int_{\text{c.s.}} \rho \mathbf{V}(\mathbf{V}\cdot\hat{\mathbf{n}})\, dA$

Momentum Equation

- **Steady Uniform Flow**

- If flow is uniform and steady, for N number of entrances and exits, the previous equation can be simplified to:

$$\Sigma \mathbf{F} = \sum_{i=1}^{N} \rho_i A_i \mathbf{V}_i (\mathbf{V}_i \cdot \hat{\mathbf{n}})$$



Horizontal nozzle with one entrance and one exit

- The momentum equation simplifies to:

$$\Sigma \mathbf{F} = \rho_2 A_2 V_2 \mathbf{V}_2 - \rho_1 A_1 V_1 \mathbf{V}_1$$

With continuity: $\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2$

$$\Sigma \mathbf{F} = \dot{m}(\mathbf{V}_2 - \mathbf{V}_1)$$

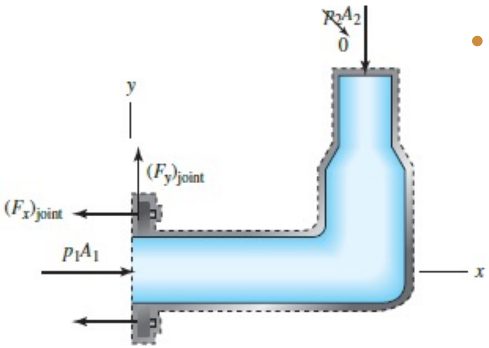
$$\Sigma F_x = \dot{m}(V_{2x} - V_{1x})$$

$$\Sigma F_y = \dot{m}(V_{2y} - V_{1y})$$

$$\Sigma F_z = \dot{m}(V_{2z} - V_{1z})$$

Momentum Equation

- **Steady Uniform Flow**



Horizontal nozzle with one entrance and one exit

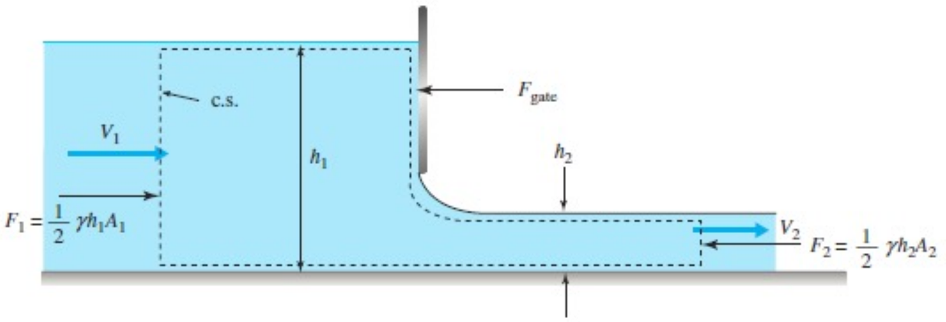
- To determine the x-component of the force of the joint on the nozzle:

$$\Sigma \mathbf{F} = \dot{m}(\mathbf{V}_2 - \mathbf{V}_1)$$

$$\Sigma F_x = -(F_x)_{\text{joint}} + p_1 A_1 = -\dot{m} V_1$$ As $(V_1)_x = V_1$ and $(V_2)_x = 0$

Momentum Equation

- **Steady Uniform Flow**



Force of the flow on a gate in a free-surface flow.

- To find the force of the gate on the flow:

$$\Sigma F_x = -F_{gate} + F_1 - F_2 = \dot{m}(V_2 - V_1)$$

Momentum Equation

Water flows through a horizontal pipe bend and exits into the atmosphere (Figure E4.11a). The flow rate is 8.3×10^{-3} m³/s. Calculate the force in each of the rods holding the pipe bend in position. Neglect body forces and viscous effects and shear force in the rods.

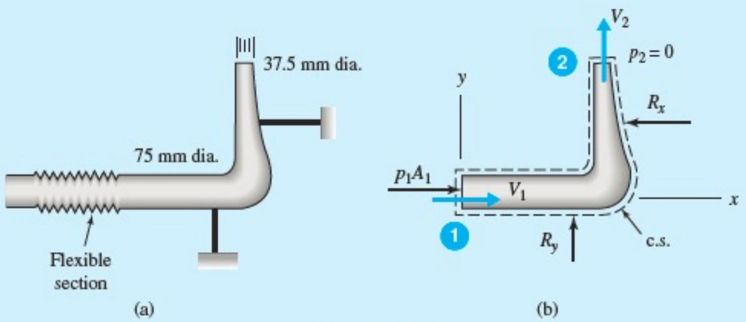


Figure E4.11

Solution

We have selected a control volume that surrounds the bend, as shown in Figure E4.11b. Since the rods have been cut, the forces that the rods exert on the control volume are included. The pressure force at the entrance of the control volume is also shown. The flexible section is capable of resisting the interior pressure, but it transmits no axial force or moment. The body force (weight of the control volume) does not act in the x- or y-direction but normal to it. Therefore, no other forces are shown. The average velocities are found to be

$$V_1 = \frac{Q}{A_1} = \frac{0.0083}{\pi\times0.075^2/4} = 1.88\text{ m/s};\quad V_2 = \frac{Q}{A_2} = \frac{0.0083}{\pi\times0.0375^2/4} = 7.52\text{ m/s}$$

Before we can calculate the forces R_x and R_y we need to find the pressures p_1 and p_2. The pressure p_2 is zero because the flow exits into the atmosphere. The pressure at section 1 can be determined using the energy equation or the Bernoulli equation. Neglecting losses between sections 1 and 2, the energy equation gives

$$\frac{V_1^2}{2g}+\frac{p_1}{\gamma} = \frac{V_2^2}{2g}+\cancelto{0}{\frac{p_2}{\gamma}}$$

$$\therefore p_1 = \frac{\gamma}{2g}(V_2^2 - V_1^2) = \frac{9810}{2\times9.81}(7.52^2 - 1.88^2) = 26.5\text{ kPa}$$

Now we can apply the momentum equation (4.6.5) in the x-direction to find R_x and in the y-direction to find R_y:

x-direction: $$p_1A_1 - R_x = \dot{m}(\cancelto{0}{V_{2x}} - V_{1x})$$

$$26\,500\times\frac{\pi}{4}\times0.075^2 - R_x = 1000\text{ kg/m}^3\times(0.0083)\text{ m}^3\text{/s }(-1.88)\text{m/s}$$

$$R_x = \underline{132.6\text{ N}}$$

y-direction: $$R_y = \dot{m}(V_{2y} - \cancelto{0}{V_{1y}})$$

$$= 1000\times(0.0083)\times7.52 = \underline{62.4\text{ N}}$$

Note that we have assumed uniform profiles and steady flow and used $\dot{m} = \rho Q$. These are the usual assumptions if information is not given otherwise.

Momentum Equation

When the velocity of a flow in an open rectangular channel of width w is relatively large, it is possible for the flow to "jump" from a depth y_1 to a depth y_2 over a relatively short distance, as shown in Figure E4.12a; this is referred to as a *hydraulic jump*. Express y_2 in terms of y_1 and V_1; assume a horizontal uniform flow.

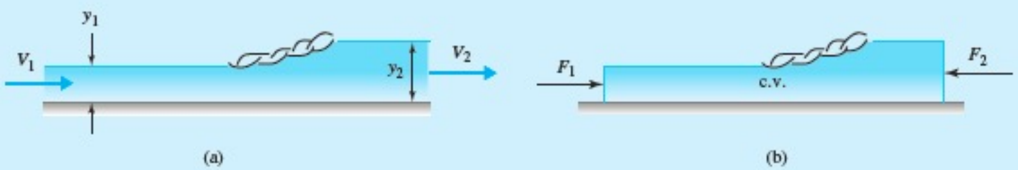


Figure E4.12

Solution

A control volume is selected as shown in Figure E4.12b with inlet and exit areas upstream and downstream of the "jump" sufficiently far that the streamlines are parallel to the wall with hydrostatic pressure distributions. Neglecting the drag that is present on the walls (if the distance between the sections is relatively small, the drag force should be negligible), the momentum equation can be manipulated as follows:

$$\Sigma F_x = \dot{m}(V_{2x} - V_{1x})$$
$$F_1 - F_2 = \rho A_1 V_1 (V_2 - V_1)$$
$$\gamma \frac{y_1}{2}(y_1 w) - \gamma \frac{y_2}{2}(y_2 w) = \rho y_1 w V_1 \left(V_1 \frac{y_1}{y_2} - V_1 \right)$$

where we have expressed F_1 and F_2 using Eq. 2.4.24, and continuity in the form of Eq. 4.4.6, so that

$$V_2 = \frac{y_1}{y_2} V_1$$

The above momentum equation can be simplified to

$$\frac{\gamma}{2}(y_1^2 - y_2^2) = \rho y_1 V_1^2 \frac{y_1 - y_2}{y_2}$$

or

$$\frac{g}{2}(y_1 - y_2)(y_1 + y_2) = \frac{y_1}{y_2} V_1^2 (y_1 - y_2)$$

The factor $(y_1 - y_2)$ is divided out and y_2 is found assuming y_1 and V_1 are known as follows:

$$\frac{g}{2}(y_1 + y_2) = \frac{y_1}{y_2} V_1^2$$
$$y_2^2 + y_1 y_2 - \frac{2}{g} y_1 V_1^2 = 0$$
$$\therefore\ y^2 = \frac{1}{2}\left(-y_1 + \sqrt{y_1^2 + \frac{8}{g} y_1 V_1^2} \right)$$

where the quadratic formula has been used. The energy equation could now be used to provide an expression for the losses in the hydraulic jump.

Momentum Equation

Consider the symmetrical flow of air around the cylinder. The control volume, excluding the cylinder, is shown in Figure E4.13. The velocity distribution downstream of the cylinder is approximated with the parabola, as shown. Determine the drag force per meter of length acting on the cylinder. Use $\rho = 1.23\ \text{kg/m}^3$.

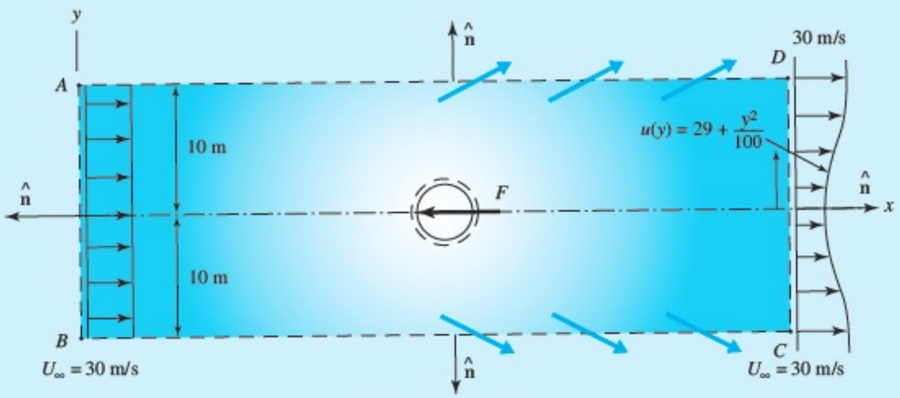


Figure E4.13

Solution

First, we must recognize that not all of the mass flux entering through AB exits through CD; consequently, some mass flux must exit AD and BC, as shown. The momentum equation (4.6.2) for the steady flow, applied to the control volume $ABCD$, takes the form

$$
\begin{aligned}
-F &= \int_{c.s} \rho V_x \mathbf{V}\cdot\hat{\mathbf{n}}\, dA = \int_{A_{CD}} \rho u\, \mathbf{V}\cdot\hat{\mathbf{n}}\, dA + \int_{A_{AD}} \rho u\, \mathbf{V}\cdot\hat{\mathbf{n}}\, dA + \int_{A_{BC}} \rho u\, \mathbf{V}\cdot\hat{\mathbf{n}}\, dA \\
&\quad + \int_{A_{AB}} \rho u\, \mathbf{V}\cdot\hat{\mathbf{n}}\, dA \\
&= \int_{A_{CD}} \rho u^2 dA + U_\infty \dot{m}_{AD} + U_\infty \dot{m}_{BC} - \int_{A_{AB}} \rho u^2 dA \\
&= 2\int_0^{10} 1.23 \left(29 + \frac{y^2}{100}\right)^2 dy + 2 \times 30 \dot{m}_{AD} - 1.23 \times 30^2 \times 20
\end{aligned}
$$

where $\dot{m}_{BC} = \dot{m}_{AD}$ is the mass flux crossing BC and AD with the x-component velocity equal to 30 m/s. The limit of 10 m was used since at $y = 10$ m, the parabola gives $u(10) = 30$ m/s. We have used Eq. 4.4.9 for $\dot{m}_{AD}$ and $\dot{m}_{BC}$ recognizing that $\mathbf{V}\cdot\hat{\mathbf{n}} = V_n$, which would be the small y-component velocity. We now use continuity to find $\dot{m}_{AD}$:

$$
\begin{aligned}
0 &= \int_{c.s} \rho\, \hat{\mathbf{n}}\cdot\mathbf{V}\, dA = \int_{A_{AD}} \rho\, \hat{\mathbf{n}}\cdot\mathbf{V}\, dA + \int_{A_{BC}} \rho\, \hat{\mathbf{n}}\cdot\mathbf{V}\, dA + \int_{A_{CD}} \rho\, \hat{\mathbf{n}}\cdot\mathbf{V}\, dA + \int_{A_{AB}} \rho\, \hat{\mathbf{n}}\cdot\mathbf{V}\, dA \\
&= \dot{m}_{AD} + \dot{m}_{BC} + 2\int_0^{10} \rho u(y)\, dy - \rho \times 20 \times 30 \\
&= 2\dot{m}_{AD} + 2\int_0^{10} 1.23 \times \left(29 + \frac{y^2}{100}\right) dy - 1.23 \times 20 \times 30
\end{aligned}
$$

$$\therefore\ \dot{m}_{AD} = 8.2 \ \text{kg/s per meter of length}$$

Evaluating the terms in the momentum equation above gives us

$$
\begin{aligned}
F &= -21170 - 492 + 22\,140 \\
&= \underline{478\ \text{N/m}}
\end{aligned}
$$

Momentum Equation

Find an expression for the head loss in a sudden expansion in a pipe in terms of V_1 and the area ratio (Figure E4.14a). Assume uniform velocity profiles and assume that the pressure at the sudden enlargement is p_1.

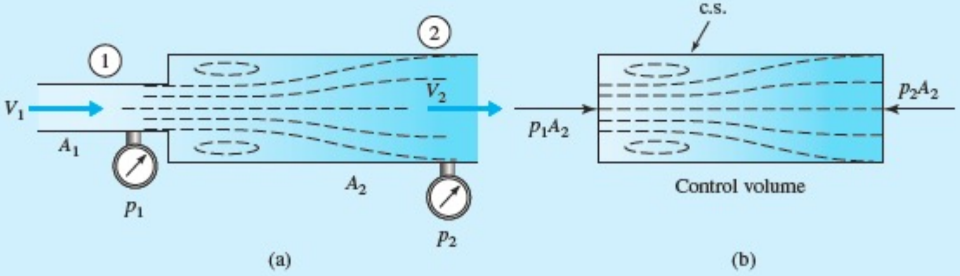


Figure E4.14

Solution

Figure E4.14a shows a sudden expansion with the diameter changing from d_1 to d_2. The pressure at the sudden enlargement is closest to p_1 since the streamlines are approximately parallel as shown (there is no pressure variation normal to parallel streamlines); they take some distance to again fill the pipe. Hence the force acting on the left end of the control volume shown in Figure E4.14b is p_1A_2. Newton's second law applied to the control volume yields, assuming uniform profiles,

$$\Sigma F_x = \dot{m}(V_2 - V_1)$$

$$(p_1 - p_2)A_2 = \rho A_2 V_2 (V_2 - V_1)$$

$$\therefore \frac{p_1 - p_2}{\rho} = V_2(V_2 - V_1)$$

The energy equation (4.5.17) provides

$$0 = \frac{V_2^2 - V_1^2}{2g} + \frac{p_2 - p_1}{\gamma} + \cancel{z_2 - z_1}^{\,0} + h_L$$

$$\therefore h_L = \frac{p_1 - p_2}{\rho g} - \frac{V_2^2 - V_1^2}{2g}$$

$$= \frac{V_2(V_2 - V_1)}{g} - \frac{(V_2 + V_1)(V_2 - V_1)}{2g} = \frac{(V_1 - V_2)^2}{2g}$$

To express this in terms of only V_1, we can use continuity and relate

$$V_2 = \frac{A_1}{A_2} V_1$$

Then the expression above for the head loss becomes

$$h_L = \left(1 - \frac{A_1}{A_2}\right)^2 \frac{V_1^2}{2g}$$

Momentum Equation

- **Momentum Equation Applied to Deflectors**

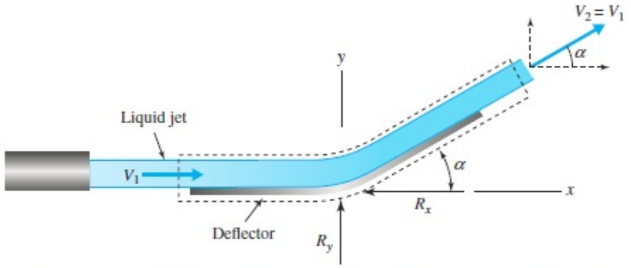
- The momentum equation is often associated with deflectors in analysis of turbomachines (turbines, pumps, and compressors).
    - There are two deflectors: fluid jets in stationary deflectors, and those deflected by moving deflectors.

Assume

- Pressure external to fluid jets is constant everywhere, so pressure in a fluid as it moves over a deflector remains constant.
- Negligible resistance due to fluid-deflector interaction.
    - Relative speed between the deflector surface and the jet stream is unchanged.
- Lateral spreading of a plane jet is neglected.
- Body force and weight of control volume is small and negligible.

Momentum Equation

- **Momentum Equation Applied to Deflectors [<u>Stationary Deflector</u>]**



A stationary deflector. (Assume no lateral spreading of the jet.)

- From Bernoulli's equation:
  - $V_2 = V_1$ (Magnitude of velocity vectors are equal)
  - Pressure is constant external to the fluid jet (and there are no changes in elevation).

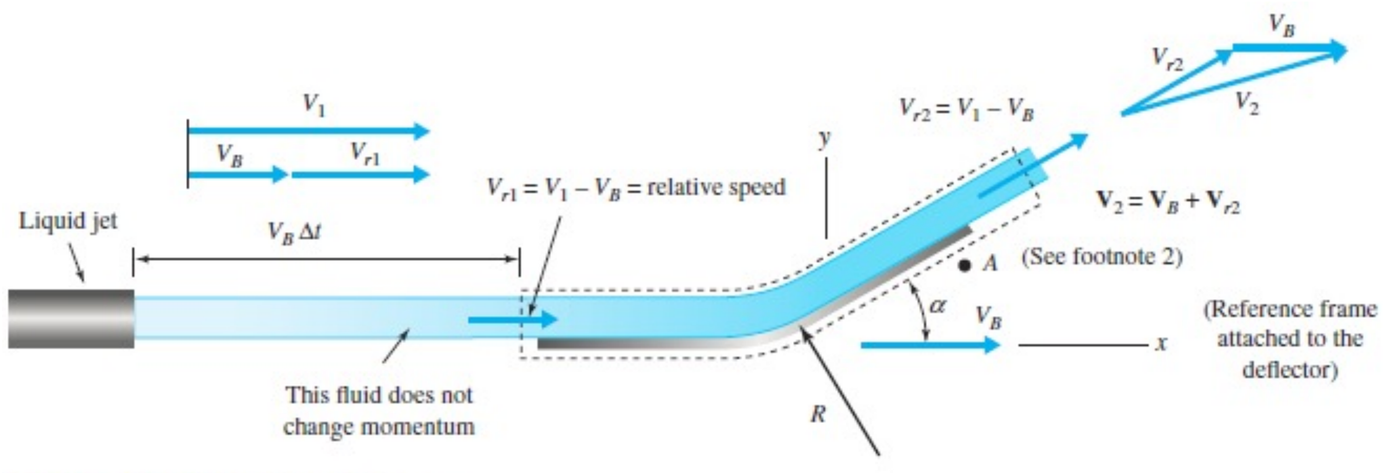
- Hence, for a steady and uniform flow, the momentum equation becomes:

$$-R_x = \dot{m}(V_2 \cos\alpha - V_1) = \dot{m}V_1(\cos\alpha - 1)$$

$$R_y = \dot{m}V_2 \sin\alpha = \dot{m}V_1 \sin\alpha$$

Momentum Equation

- **Momentum Equation Applied to Deflectors [<u>Moving Deflectors</u>]**



A moving deflector.

- Can be either a single moving deflector (snowplow blade) or a series of moving deflectors (turbine vanes).

Momentum Equation

- **Momentum Equation Applied to Deflectors [<u>Moving Deflectors</u>]**

- For a single deflector moving in the positive x-direction with the speed V_B.
- In a reference frame attached to a stationary nozzle, the flow is unsteady.
  - However, in a reference frame attached to the deflector, a steady flow is observed.

- Hence, the momentum equation becomes:

$$-R_x = \dot{m}_r (V_1 - V_B)(\cos\alpha - 1)$$
$$R_y = \dot{m}_r (V_1 - V_B)\sin\alpha$$

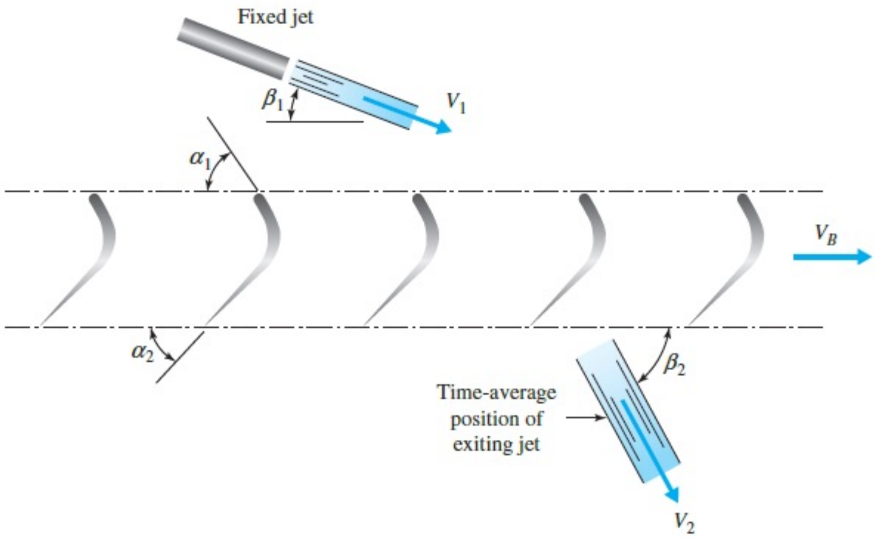
- Where $\dot{m}_r$ is the mass flux exiting the fixed jet with a changed momentum.

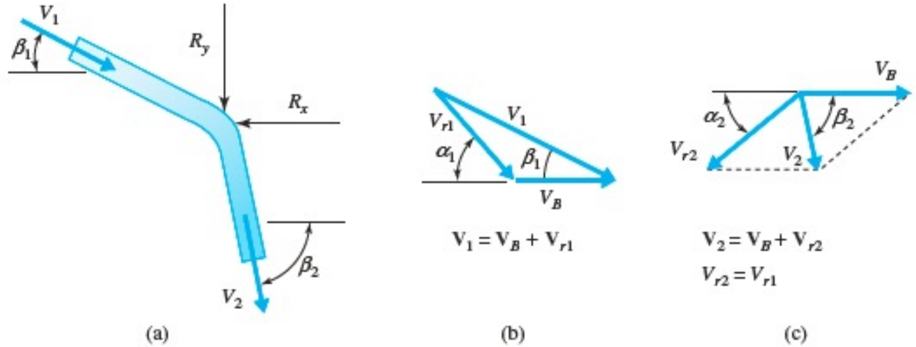
$$\dot{m}_r = \rho A (V_1 - V_B)$$

Momentum Equation

- **Momentum Equation Applied to Deflectors [<u>Moving Deflectors</u>]**

- For a cascade of vanes:

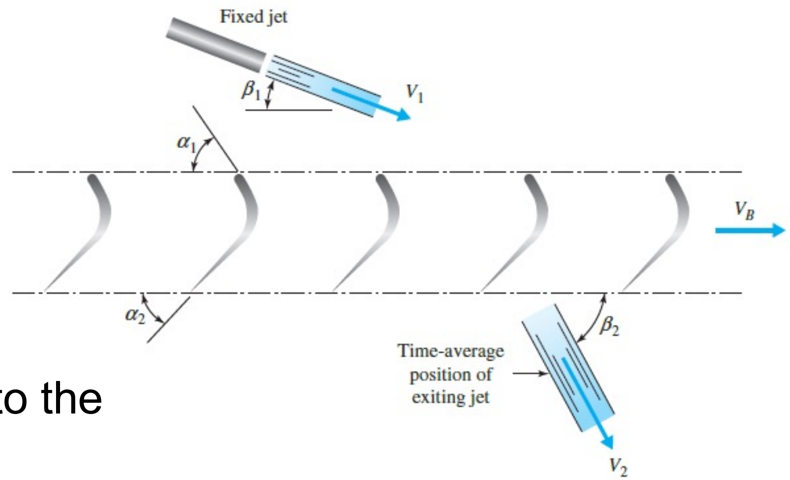




Detail of the flow situation involving a series of vanes: (a) average position of jet; (b) entrance velocity polygon; (c) exit velocity polygon.

Momentum Equation

- **Momentum Equation Applied to Deflectors [<u>Moving Deflectors</u>]**



- If all of the mass exiting the jet has a changed momentum:

$$-R_x = \dot{m}(V_{2x} - V_{1x})$$

- As the x-component of the force is related to the power output:
  - Found by multiplying this force by the blade speed for each jet (N).

$$\dot{W} = NR_xV_B$$

- Y-component force doesn't move in the y-direction: produces no power.

Momentum Equation

A deflector turns a sheet of water through an angle of 30° as shown in Figure E4.15. What force components are necessary to hold the deflector in place if $\dot{m} = 32$ kg/s?

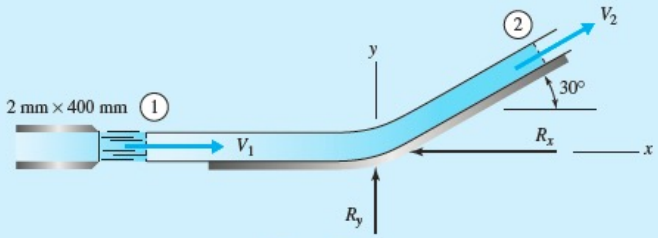


Figure E4.15

Solution

The control volume we have selected includes the deflector and the water adjacent to it. The only force that is acting on the control volume is due to a support needed to hold the deflector. This force has been decomposed into R_x and R_y.

The velocity V_1 is found to be

$$V_1 = \frac{\dot{m}}{\rho A_1}$$

$$= \frac{32\,\text{kg/s}}{1000\,\text{kg/m}^3 \times (0.002 \times 0.4)\text{m}^2} = 40\text{ m/s}$$

Bernoulli's equation (3.4.8) shows that if the pressure does not change, then the magnitude of the velocity does not change, provided that there is no significant change in elevation and that viscous effects are negligible; thus we can conclude that $V_2 = V_1$ since $p_2 = p_1$. Next, the momentum equation is applied in the x-direction to find R_x and then in the y-direction for R_y:

x-direction:

$$-R_x = \dot{m}(V_{2x} - V_{1x})$$

$$= 32\text{ kg/s}(40\cos 30° - 40)\text{m/s}$$

$$\therefore R_x = \underline{172\text{ N}}$$

y-direction:

$$R_y = \dot{m}(V_{2y} - \cancel{V_{1y}}^{\,0})$$

$$= 32\,(40\sin 30°) = \underline{640\text{ N}}$$

Momentum Equation

The deflector shown in Figure E4.16 moves to the right at 30 m/s while the nozzle remains stationary. Determine (a) the force components needed to support the deflector, (b) $\mathbf{V}_2$ as observed from a fixed observer, and (c) the power generated by the vane. The jet velocity is 80 m/s.

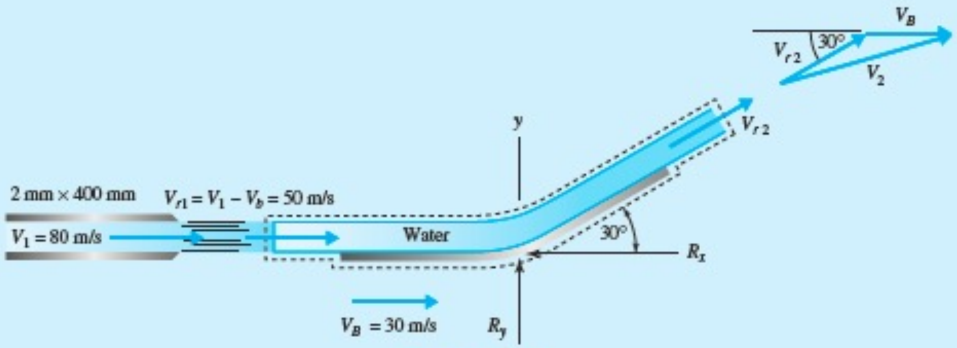


Figure E4.16

Solution

(a) To solve the problem of a moving deflector, we observe the flow from a reference frame attached to the deflector. In this moving reference frame the flow is steady and Bernoulli's equation with $p_1 = p_2$ can then be used to show that $V_{r1} = V_{r2} = 50$ m/s, the velocity of the sheet of water as observed from the deflector. Note that we cannot apply Bernoulli's equation in a fixed reference frame since the flow would not be steady. Applying the momentum equation to the moving control volume, which is indicated again by the dashed line, we obtain the following:

x-direction:
$$-R_x = \dot{m}_r\,[(V_{r2})_x - (V_{r1})_x]$$
$$= 1000\ \text{kg/m}^3 \times 0.002\ \text{m} \times 0.4\ \text{m} \times 50\ \text{m/s}\,(50\cos 30^\circ - 50)\ \text{m/s}$$
$$\therefore R_x = \underline{268\ \text{N}}$$

y-direction:
$$R_y = \dot{m}_r\,[(V_{r2})_y - \cancelto{0}{(V_{r1})_y}]$$
$$= 1000 \times 0.002 \times 0.4 \times 50(50\sin 30^\circ) = \underline{1000\ \text{N}}$$

When calculating $\dot{m}$, we must use only that water which has its momentum changed; hence the velocity used is 50 m/s.

(b) Observed from a fixed observer the velocity $\mathbf{V}_2$ of the fluid after the deflection is $\mathbf{V}_2 = \mathbf{V}_{r2} + \mathbf{V}_B$, where $\mathbf{V}_{r2}$ is directed tangential to the deflector at the exit and has a magnitude equal to $\mathbf{V}_{r1}$ (see the velocity diagram above). Thus

$$(V_2)_x = V_{r2}\cos 30^\circ + V_B$$
$$= 50 \times 0.866 + 30 = 73.3\ \text{m/s}$$
$$(V_2)_y = V_{r2}\sin 30^\circ$$
$$= 50 \times 0.5 = 25\ \text{m/s}$$

Finally,

$$\mathbf{V}_2 = \underline{73.3\hat{\mathbf{i}} + 25\hat{\mathbf{j}}\ \text{m/s}}$$

(c) The power generated by the moving vane is equal to the velocity of the vane times the force the vane exerts in the direction of the motion. Therefore,

$$\dot{W} = V_B \times R_x = 30\ \text{m/s} \times 268\ \text{N} = \underline{8040\ \text{W}}$$

Momentum Equation

High-speed air jets strike the blades of a turbine rotor tangentially while the 1.5-m-diameter rotor rotates at 140 rad/s (Figure E4.17a). There are 10 such 40-mm-diameter jets. Calculate the maximum power output. The air density is 2.4 kg/m³. Neglect any lateral spreading.

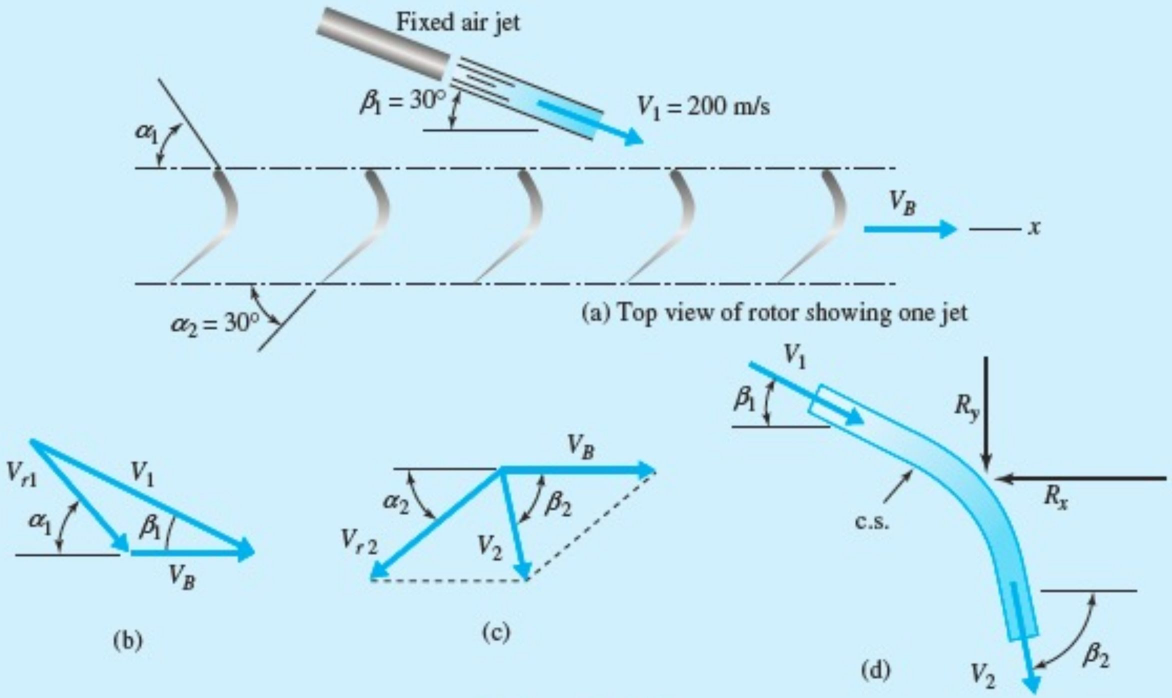


Figure E4.17

Momentum Equation

Solution

The blade angle α_1 is set by demanding that the air jet enter the blades tangentially, as observed from the moving blade; that is, the relative velocity vector $\mathbf{V}_r$ must make the angle α_1 with respect to the blade velocity $\mathbf{V}_B$. This is shown in Figure E4.17b. The relative entrance velocity is $\mathbf{V}_{r1}$ (Figure E4.17b), and the relative exit velocity is $\mathbf{V}_{r2}$ (Figure E4.17c). Both velocity polygons are presented by the vector equation

$$\mathbf{V} = \mathbf{V}_r + \mathbf{V}_B$$

which states that the absolute velocity equals the relative velocity plus the blade velocity. From the polygon at the entrance we have

$$\begin{aligned} V_1 \sin\beta_1 &= V_{r1} \sin\alpha_1 \\ V_1 \cos\beta_1 &= V_{r1} \cos\alpha_1 + V_B \\ \therefore\ 200\ \sin 30° &= V_{r1} \sin\alpha_1 \\ 200\ \cos 30° &= V_{r1} \cos\alpha_1 + 0.75 \times 140 \end{aligned}$$

where $\mathbf{V}_B$ is the radius multiplied by the angular velocity. A simultaneous solution yields

$$V_{r1} = 121\ \text{m/s} \qquad \alpha_1 = 55.7°$$

The friction between the air and the blade is quite small and can be neglected when calculating the maximum output. This allows us to assume $V_{r2} = V_{r1}$. From the exiting velocity polygon we can write

$$\begin{aligned} V_B - V_{r2} \cos\ \alpha_2 &= V_2 \cos\ \beta_2 \\ V_{r2} \sin\ \alpha_2 &= V_2 \sin\ \beta_2 \\ \therefore\ 0.75 \times 140 - 121\ \cos\ 30° &= V_2 \cos\ \beta_2 \\ 121\ \sin 30° &= V_2 \sin\ \beta_2 \end{aligned}$$

A simultaneous solution results in

$$V_2 = 60.5\ \text{m/s} \qquad \beta_2 = 89.8°$$

The momentum equation applied to the control volume, shown in Figure E4.17d, gives

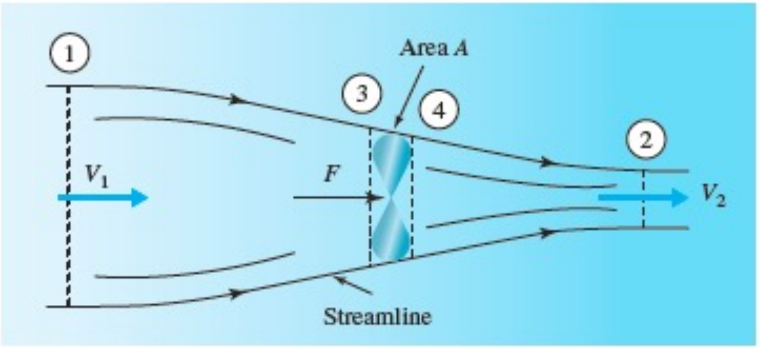
$$\begin{aligned} -R_x &= \dot{m}(V_{2x} - V_{1x}) \\ &= 2.4\ \text{kg/m}^3 \times \pi \times 0.02^2\ \text{m}^2 \times 200\ \text{m/s}(60.5 \cos 89.8° - 200 \cos 30°)\ \text{m/s} \\ \therefore\ R_x &= 104.3\ \text{N} \end{aligned}$$

There are 10 jets, each producing the force above. The maximum power output is then

$$\begin{aligned} \text{power} &= 10 \times R_x \times V_B \\ &= 10 \times 104.3\ \text{N} \times (0.75 \times 140)\ \text{m/s} = 109\,600\ \text{W} \quad \text{or} \quad \underline{109.6\ \text{kW}} \end{aligned}$$

Momentum Equation

- **Momentum Equation Applied to Propellers**



A propeller in a fluid flow.

- The momentum equation in a control volume between sections 1 and 2 gives:

$$F = \dot{m}(V_2 - V_1)$$

- Since the areas A_1 and A_2 are not known, another control volume is drawn close to the propeller.
  - $V_3 \approx V_4$ and $A_3 \approx A_4 = A$

$$F + p_3 A - p_4 A = 0$$

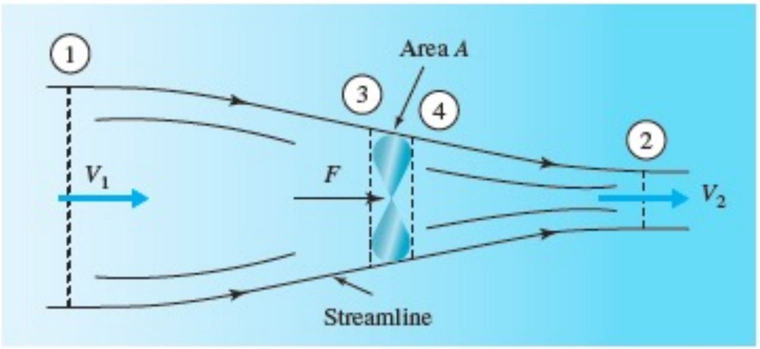
- Neglecting viscous effects, and realizing that $P_1 = P_2 = P_{atm}$

$$\frac{V_1^2 - V_3^2}{2} + \frac{p_1 - p_3}{\rho} = 0 \quad \text{and} \quad \frac{V_4^2 - V_2^2}{2} + \frac{p_4 - p_2}{\rho} = 0$$

$$V_3 = \frac{1}{2}(V_2 + V_1)$$

Momentum Equation

- **Momentum Equation Applied to Propellers**



A propeller in a fluid flow.

- The power input for this propeller is then:

$$\dot{W}_{\text{fluid}} = \frac{V_2^2 - V_1^2}{2}\dot{m}$$

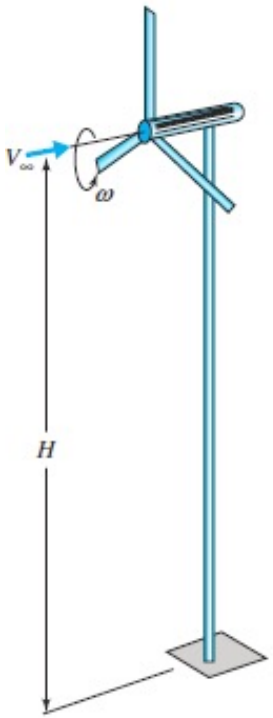
- The moving propeller requires power:

$$\begin{aligned}\dot{W}_{\text{prop}} &= F \times V_1 \\ &= \dot{m}(V_2 - V_1)V_1\end{aligned}$$

- Theoretical propeller efficiency: $\eta_P = \dfrac{\dot{W}_{\text{prop}}}{\dot{W}_{\text{fluid}}} = \dfrac{V_1}{V_3}$

Momentum Equation

- **Control-Volume Analysis of Wind Turbines**



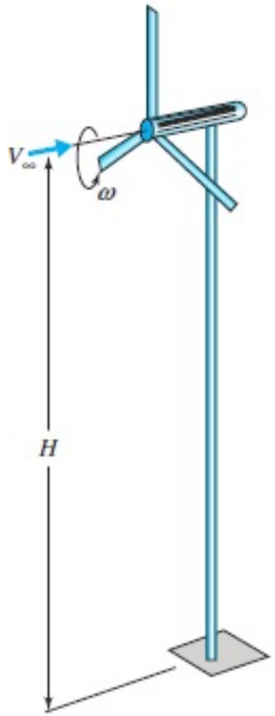
A typical horizontal-axis wind turbine.

- A wind turbine extracts energy from the airflow.
- Velocity downstream is reduced.
- Diameter downstream is increased.
- Convert wind power to mechanical power (and then to electric power through electric generators).

- Two main types: Vertical-axis and horizontal-axis wind turbines.

Momentum Equation

- **Control-Volume Analysis of Wind Turbines**



A typical horizontal-axis wind turbine.

- Power in a wind stream is equal to the rate of incoming kinetic energy of the wind within a streamtube.

$$\dot{W} = \frac{1}{2}\dot{m}V_1^2$$

With V_1 being the upstream wind velocity, and $\dot{m}$ is the mass flow rate through the streamtube.

$$\dot{m} = \rho A_1 V_1$$

- Hence, the available power for this turbine is:

$$\dot{W} = \frac{1}{2}\rho A_1 V_1^3$$

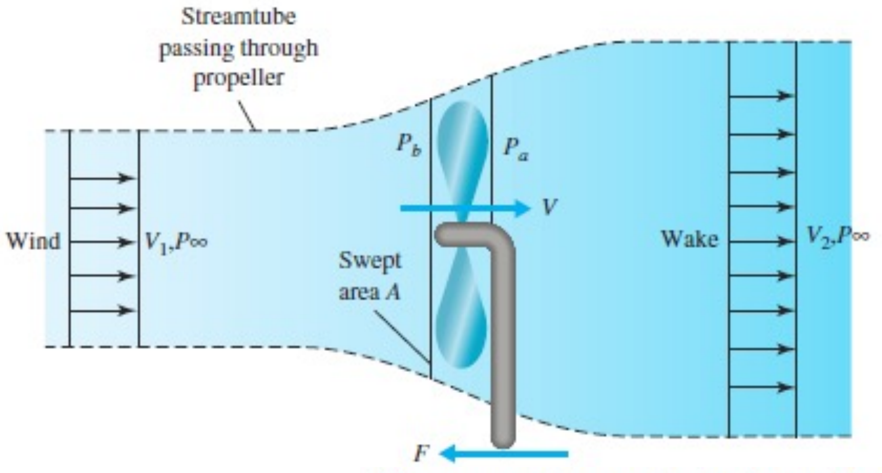
- The total energy, E, that is found by integrating the above equation over a time period:

$$E = \int_0^t \dot{W}dt = \frac{1}{2}\rho A \int_0^t V^3 dt$$

Momentum Equation

- **Control-Volume Analysis of Wind Turbines**

Between a and b, the velocity of wind decreases as the kinetic energy of the wind is converted to mechanical work.



The streamtube containing the turbine.

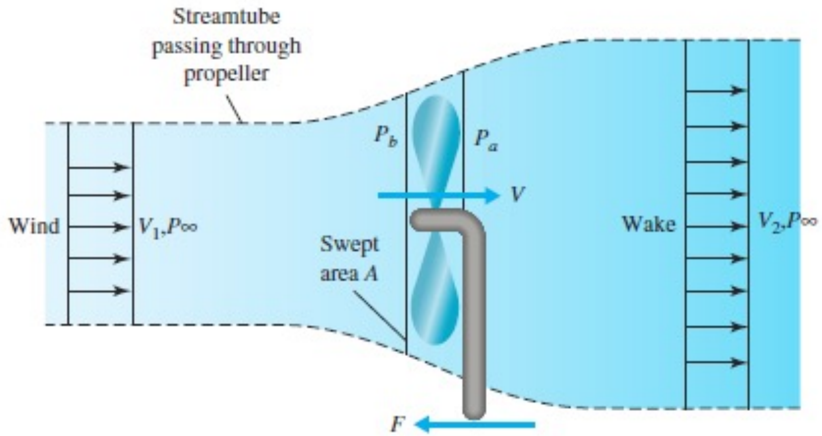
- Between a section a distance upstream of the turbine (1) and at the inlet of the turbine (a), Bernoulli's equation is used:

$$p_1 + \frac{1}{2}\rho V_1^2 = p_a + \frac{1}{2}\rho V_a^2$$

Assume:
An adiabatic, incompressible flow
Negligible changes in potential energy

Momentum Equation

- **Control-Volume Analysis of Wind Turbines**



The streamtube containing the turbine.

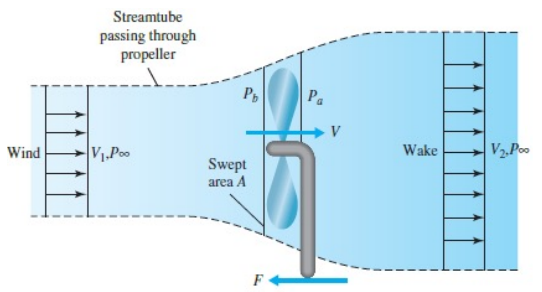
- Between a section a distance downstream of the turbine (2) and section b:

$$p_b + \frac{1}{2}\rho V_b^2 = p_2 + \frac{1}{2}\rho V_2^2$$

Assume:
An adiabatic, incompressible flow
Negligible changes in potential energy

Momentum Equation

- **Control-Volume Analysis of Wind Turbines**

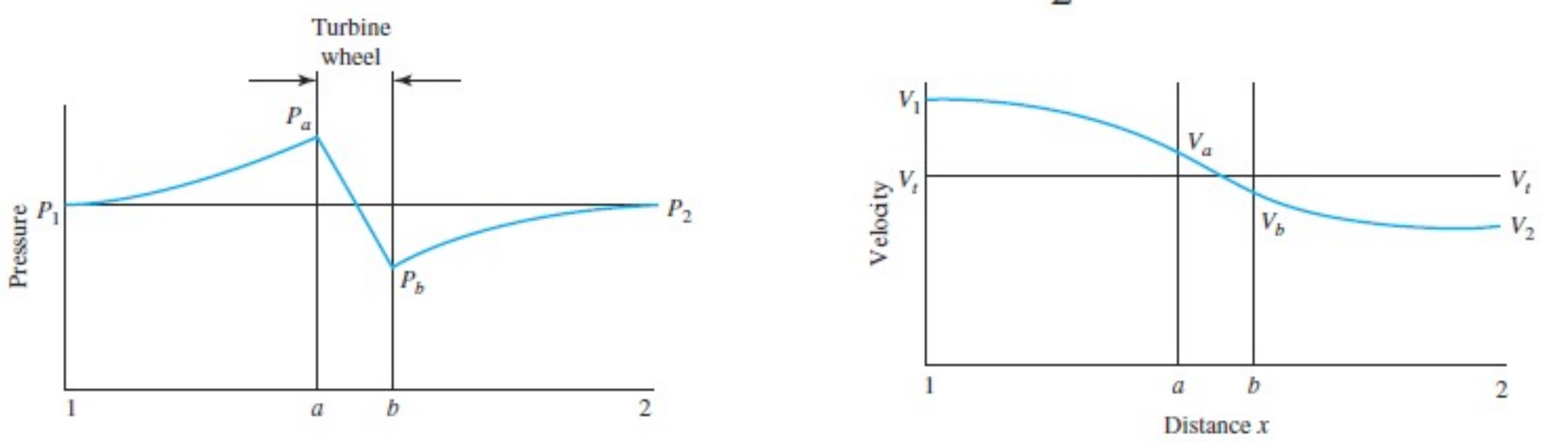


- The pressure difference across the turbine (a and b):

$$p_a - p_b = (p_1 - p_2) + \frac{1}{2}\rho(V_1^2 - V_2^2) + \frac{1}{2}\rho(V_b^2 - V_a^2)$$

  - Between points 1 and 2, the pressure is at ambient $p_1 = p_2 = p_\infty$
  - The change in wind velocity across the turbine is negligible.

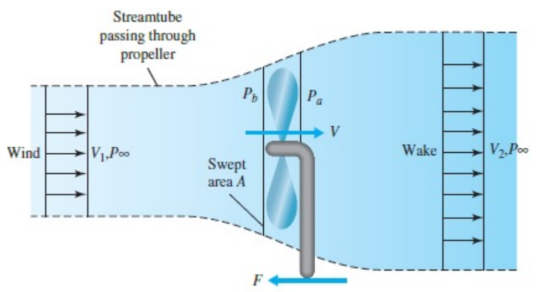
$$p_a - p_b = \frac{1}{2}\rho(V_1^2 - V_2^2)$$



Representative pressure and velocity profiles across a horizontal-axis turbine.

Momentum Equation

- **Control-Volume Analysis of Wind Turbines**



- The Force, F, on the turbine support is found using the momentum equation across sections 1 and 2.

$$-F = \dot{m}(V_2 - V_1)$$

- Between sections a and b:

$$-F + (p_a - p_b)A = \dot{m}(V_b - V_a)$$

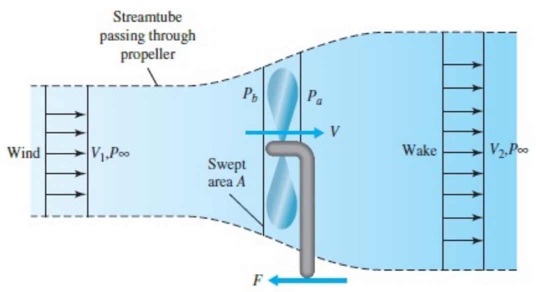
  - A is the swept area of the turbine blades, and $\dot{m} = \rho V_t A$, with V_t being the average wind velocity across the turbine.

- Hence, the wind power is:

$$\dot{W} = \frac{1}{2}\rho A V_t(V_1^2 - V_2^2) \qquad \dot{W} = \frac{1}{4}\rho A(V_1 + V_2)(V_1^2 - V_2^2)$$

Momentum Equation

- **Control-Volume Analysis of Wind Turbines**



- The fluid cannot come to a complete stop at the turbine exit.
- There is an optimum fluid exit velocity that results in maximum power.

$$V_2 = \frac{1}{3} V_1$$

- Hence, the maximum power that can be extracted from the wind turbine is:

$$\dot{W}_{\max} = \frac{8}{27} \rho A V_1^3$$

- The maximum efficiency (power coefficient) of the turbine is calculated as follows:

$$\eta_{\max} = \frac{\dot{W}_{\max}}{\frac{1}{2}\rho A V_1^3} = \frac{\frac{8}{27}\rho A V_1^3}{\frac{1}{2}\rho A V_1^3} = 0.593 \quad \text{or} \quad 59.3\%$$

Momentum Equation

A wind turbine with a rotor diameter of 50 m and is exposed to a 16 km/h wind at standard atmospheric conditions (100 kPa, 25°C). Calculate the following:

(a) The available wind power

(b) The maximum power that can be obtained from the wind

(c) The maximum torque if the turbine rotates at 40 rpm

Solution

(a) The available power is $\dot{W} = \frac{1}{2}\dot{m}V_1^2$ or $\dot{W} = \frac{1}{4}\rho A V_1^3$. The density of the air is calculated as

$$\rho = \frac{p}{RT} = \frac{100 \times 10^3 \times 29}{8314 \times 298} = 1.17 \text{ kg/m}^3$$

The area swept by the rotor is $A = \pi(25)^2 = 1963\,\text{m}^2$. The wind velocity is

$$V_1 = \frac{1.6 \times 10^3}{3600} = 4.44 \text{ m/s}$$

and the available power is

$$\dot{W} = \frac{1}{2}(1.17 \text{ kg/m}^3 \times 1963\,\text{m}^2)(4.47 \text{ m/s})^3 = 103\,000 \text{ W} \quad \text{or} \quad \underline{103 \text{ kW}}$$

(b) The maximum power that could be obtained is

$$\dot{W}_{\text{max}} = \eta \dot{W}_{\text{available}} = 0.593 \times 103 = \underline{61.1 \text{ kW}}$$

(c) Torque is related to power by $\dot{W} = 2\pi n T$ if n is measured in rpm. Then

$$T = \frac{\dot{W}_{\text{max}}}{2\pi n} = \frac{61.1 \times 10^3 \text{ N}\cdot\text{m/s}}{(2\pi \text{ rad/rev}) \times 40 \text{ rev/60 s}} = \underline{14\,600 \text{ N}\cdot\text{m}}$$

Momentum Equation

The wind turbine considered in the previous example operates in a wind that has a velocity pattern during a day that could be approximated by half a sine wave as $V = V_m \sin(\pi t/24)$, where t is time in hours and $V_m = 8.5$ m/s. Calculate:

(a) The theoretical maximum wind turbine energy produced during a 24-hr period

(b) The wind mean energy velocity

(c) The electric energy generated during a 24-hr period if the overall efficiency of the wind turbine-generator is 0.3 (the overall efficiency includes efficiency of the generator and friction losses in the gear system used to couple the wind turbine to the generator)

(d) If the wind turbine has a cut-in velocity of 3.1 m/s and a cut-out velocity of 8 m/s, calculate the theoretical maximum wind turbine energy during a 24-hr period.

Solution

(a) The total wind energy during a 24-hr period is

$$E = \frac{1}{2}\rho A \int_0^{24} V^3\, dt = \frac{1}{2}\rho A \int_0^{24} [V_m \sin(\pi t/24)]^3\, dt$$

Integrating we get

$$E = \frac{1}{2}\rho A V_m^3 \left[\frac{\cos^3(\pi t/24)}{\pi/8} - \frac{\cos(\pi t/24)}{\pi/24}\right]_0^{24} = \frac{1}{2}\rho A V_m^3\left(\frac{32}{\pi}\right)$$

Using the density and area from previous example and $t = 24$ hrs, we calculate the energy

$$E = \frac{1}{2}(1.17\,\text{kg/m}^3 \times 1963\,\text{m}^2)(8.5\,\text{m/s})^3\left(\frac{32}{\pi}\text{hr}\right) = \underline{7180\ \text{kW-hr}}$$

The maximum wind energy is $E_{max} = \eta_{max} \times E = 0.593 \times 7180 = \underline{4260\ \text{kW-hr}}$

(b) The wind mean energy velocity can be calculated using

$$V_{mean} = \left(\frac{E_{max}}{\frac{1}{2}\rho A\,\Delta t}\right)^{1/3} = \left(\frac{4260 \times 10^3}{\frac{1}{2}(1.17 \times 1963 \times 24)}\right)^{1/3} = \underline{5.37\ \text{m/s}}$$

(c) The total electric energy generated during the 24-hr period is

$$E_e = \eta_e \times E_{max} = 0.3 \times 4260 = 1280\ \text{kW-hr}$$

Momentum Equation

(d) In this case, we have to determine the time during the day when the wind turbine is operational, that is the wind speed is between the cut-in and cut-out velocities. Using the given expression for V we calculate the time during the day when $V = 3.1\,\text{m/s}$ as

$$t_1 = \frac{24}{\pi}\sin^{-1}(3.1/8.5) = 2.85\,\text{hr}$$

where $\sin^{-1}(3.1/8.5) = 0.372$ rad. Similarly, we calculate t_2 when $V = 8$ m/s as

$$t_2 = \frac{24}{\pi}\sin^{-1}(8/8.5) = 9.4\,\text{hr}$$

To determine the total energy during the day we integrate between the above two times. Note that since the wind velocity pattern is symmetrical with respect to the 12-h during the day, the cut-in and cut-out times occur twice during the day. Hence, we calculate the energy as follows:

$$E = \frac{1}{2}\rho A\left(2 \times \int_{t_1}^{t_2} V^3\,dt\right) = \frac{1}{2}\rho A\left(2 \times \int_{t_1}^{t_2}\left[V_m \sin(\pi t/24)\right]^3 dt\right)$$

$$= \frac{1}{2}\rho A V_m^3\left[\frac{\cos^3(\pi t/24)}{\pi/8} - \frac{\cos(\pi t/24)}{\pi/24}\right]_{t_1}^{t_2} = \frac{1}{2}\rho A V_m^3(2 \times 2.602) = \underline{3702\ \text{kW-hr}}$$

The maximum wind energy is

$$E_{\max} = \eta_{\max} \times E = 0.593 \times 3702 = \underline{2190\ \text{kW-hr}}$$

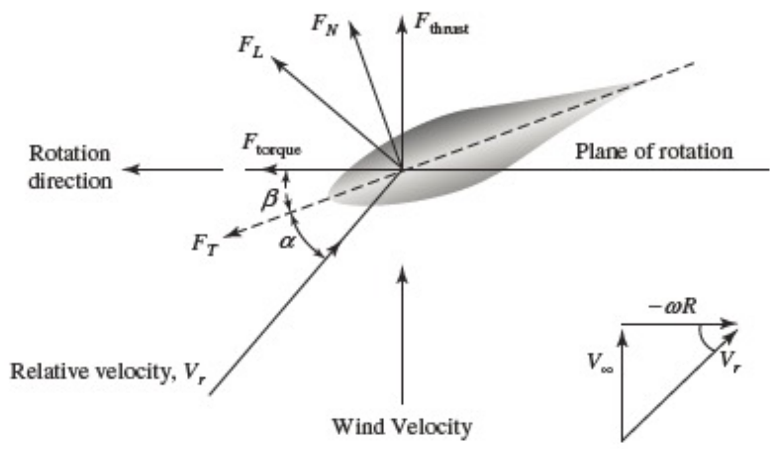
The electric power generated is

$$E_e = \eta_e \times E_{\max} = 0.3 \times 2190 = \underline{658\ \text{kW-hr}}$$

In order to protect the wind turbine, generator, and gearing system from wear and high forces, the wind turbine is designed to shut down when the wind velocity is below the cut-in velocity and above the cut-out velocity.

Momentum Equation

- **Aerodynamic Analysis of a Rotating Wind-Turbine Blade**



A rotating wind-turbine blade and a velocity triangle.

- More detail into blade geometry is needed as the flow through the wind turbine is dependent on the airfoil.

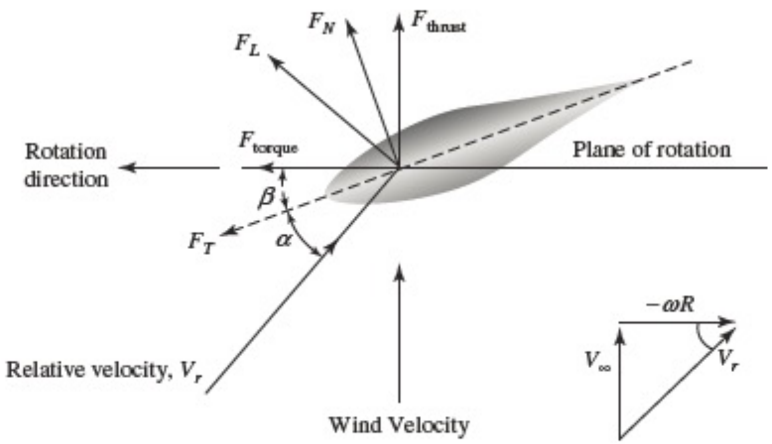
  Blade Profile characteristics
  - ω: Angular speed
  - α: Angle of attack of the blade w.r.t. relative velocity of the wind
  - β: Pitch angle between the chord line and the plane of rotation of the rotor.

- The lift and drag forces are calculated using the relative velocity.
  - This vector is the velocity of the wind relative to the airfoil.

$$\mathbf{V}_r = \mathbf{V}_\infty - \mathbf{V}_B$$

Momentum Equation

- **Aerodynamic Analysis of a Rotating Wind-Turbine Blade**



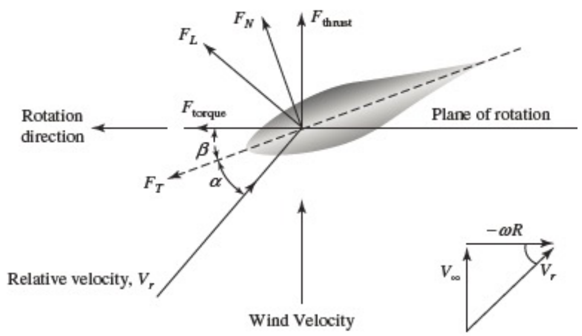
A rotating wind-turbine blade and a velocity triangle.

- From previous, the blade velocity is $V_B = \omega R$, with ω being the angular velocity of the rotor and R the distance between the center of rotation and the position on the airfoil where the forces are determined.
- V_∞ is the free-stream velocity.
- The relative velocity (from the triangle) is:

$$V_r = \frac{V_\infty}{\sin(\alpha + \beta)}$$

Momentum Equation

- **Aerodynamic Analysis of a Rotating Wind-Turbine Blade**



A rotating wind-turbine blade and a velocity triangle.

- A two-dimensional approach to find the lift/drag (per unit length) is:

$$\text{Lift force:} \quad F_L = \frac{1}{2} C_L \rho V_r^2 c$$

$$\text{Drag Force:} \quad F_D = \frac{1}{2} C_D \rho V_r^2 c$$

  - c: Chord length (distance from the nose to the tail)
  - C_L and C_D are lift and drag coefficients

- The normal/tangential forces and the coefficients are:

$$F_N = F_L \cos\alpha + F_D \sin\alpha$$
$$F_T = F_L \sin\alpha - F_D \cos\alpha$$

$$C_N = \frac{F_N}{\frac{1}{2}\rho V_r^2 c} \qquad C_T = \frac{F_T}{\frac{1}{2}\rho V_r^2 c}$$

$$C_N = C_L \cos\alpha + C_D \sin\alpha$$
$$C_T = C_L \sin\alpha - C_D \cos\alpha$$

Momentum Equation

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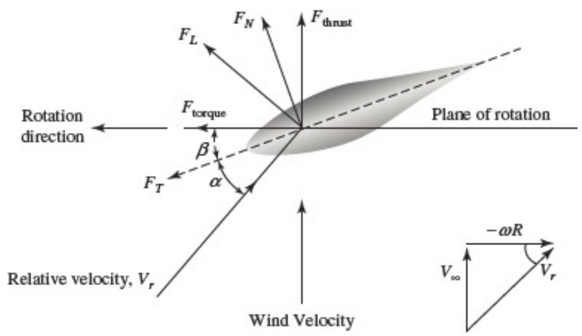

A rotating wind-turbine blade and a velocity triangle.

- The thrust/torque forces for a rotating blade and the respective coefficients are found as:

$$F_{\text{thrust}} = F_L \cos(\alpha + \beta) + F_D \sin(\alpha + \beta)$$
$$F_{\text{torque}} = F_L \sin(\alpha + \beta) - F_D \cos(\alpha + \beta)$$

$$C_{\text{thrust}} = \frac{F_{\text{thrust}}}{\frac{1}{2}\rho V_r^2 c} \qquad C_{\text{torque}} = \frac{F_{\text{torque}}}{\frac{1}{2}\rho V_r^2 c}$$

$$C_{\text{thrust}} = C_L \cos(\alpha + \beta) + C_D \sin(\alpha + \beta)$$
$$C_{\text{thrust}} = C_L \sin(\alpha + \beta) - C_D \cos(\alpha + \beta)$$

- The torque (per unit length) on the blade is $T = F_{\text{torque}} \times R$

- The total torque is obtained by multiplying the torque of a single blade by the number of blades. Hence the power:

$$P = N \times 2\pi n T$$

N: Number of blades
n: Rotational speed (rev/s)

Momentum Equation

A wind at 1 atm and 15°C is blowing at 20 m/s over a 10-m-diameter wind turbine that has a cutout velocity of 15 m/s. In this case, since the wind velocity is higher than the cutout velocity the rotor is "parked" (the rotor is locked so it is stationary) to prevent damage to the wind turbine. At a location midway between the root and tip of the blade, the chord length is 0.46 m, the angle of attack is 14°, and the lift and drag coefficients are 0.93 and 0.74, respectively. Calculate the normal and tangential forces acting at that location.

Solution

Since the rotor is stationary the relative velocity is equal to the free-stream wind velocity, that is, $V_r = V_\infty = 20$ m/s. The air density is

$$\rho = \frac{p}{RT} = \frac{101}{0.287 \times 288} = 1.23\,\text{kg/m}^3$$

The lift and drag forces are calculated using

$$F_L = \frac{1}{2} C_L \rho V_r^2 c = \frac{0.93}{2} \times (1.23\,\text{kg/m}^3) \times (20\,\text{m/s})^2 \times 0.46\,\text{m} = 105\,\text{N/m}$$
$$F_D = \frac{1}{2} C_D \rho V_r^2 c = \frac{0.74}{2} \times (1.23\,\text{kg/m}^3) \times (20\,\text{m/s})^2 \times 0.46\,\text{m} = 83.7\,\text{N/m}$$

The normal and tangential forces are

$$F_N = F_L \cos\alpha + F_D \sin\alpha = 105\cos 14 + 83.7\sin 14 = \underline{122\,\text{N/m}}$$
$$F_T = F_L \sin\alpha - F_D \cos\alpha = 105\sin 14 - 83.7\cos 14 = \underline{-55.8\,\text{N/m}}$$

The minus sign indicates that the force is acting in a direction opposite to that shown in Figure 4.21. Note that in this case, $F_{\text{thrust}} = F_D$ and $F_{\text{torque}} = F_L$.

Momentum Equation

A three-bladed rotor wind turbine has a diameter of 10 m, is operating in a 7 m/s wind at 1 atm and 25°C. The rotational speed is 72 rpm and the chord length is 0.46 m. At a location on the rotor blade where the radial distance is 50% of the blade length, the pitch angle is 9°, and the tangential and normal coefficients are 0.09 and 0.83, respectively. If the blade has a uniform chord length, airfoil profile, and pitch angle, calculate the torque and thrust force acting at that location. Then make an estimate of the torque and the power produced assuming that the average value acts over the entire length of the blade.

Solution

Since this is a rotating blade we calculate, using $\omega = 2\pi$ rad/rev $\times$ 1.2 rev/s = 7.54 rad/s and $R = 0.5 \times 5 = 2.5$ m, the relative velocity of the wind is

$$V_r = \sqrt{V_\infty^2 + (\omega R)^2} = \sqrt{49 + (7.54 \times 2.5)^2} = 20 \text{ m/s}$$

The local angle of attack is calculated using

$$\alpha = \sin^{-1}(V_\infty / V_r) - \beta = \sin^{-1}(7/20) - 9° = 11.5°$$

We can determine the lift and drag coefficients by manipulating Eqs. 4.6.49. The result is

$$C_L = C_N \cos\alpha + C_T \sin\alpha$$
$$C_D = C_N \sin\alpha - C_T \cos\alpha$$

Momentum Equation

The lift and drag coefficients can then be calculated to be

$$C_L = 0.83\ \cos 11.5 + 0.09\ \sin\ 11.5 = 0.831$$
$$C_D = 0.83\ \sin 11.5 - 0.09\ \cos\ 11.5 = 0.077$$

Next, we calculate the thrust and torque coefficients:

$$C_{\text{thrust}} = C_L \cos(\alpha + \beta) + C_D \sin(\alpha + \beta) = 0.831\ \cos 20.5 + 0.077\ \sin 20.5 = 0.805$$
$$C_{\text{torque}} = C_L \sin(\alpha + \beta) - C_D \cos(\alpha + \beta) = 0.831\ \sin 20.5 - 0.077\ \cos 20.5 = 0.219$$

The forces are calculated as follows

$$F_{\text{thrust}} = \frac{1}{2} C_{\text{thrust}} \rho V_r^2 c = \frac{0.805}{2} \times (1.23\ \text{kg/m}^3) \times (20\ \text{m/s})^2 \times 0.46\ \text{m} = 91\ \text{N/m}$$
$$F_{\text{torque}} = \frac{1}{2} C_{\text{torque}} \rho V_r^2 c = \frac{0.219}{2} \times (1.23\ \text{kg/m}^3) \times (20\ \text{m/s})^2 \times 0.46\ \text{m} = 25\ \text{N/m}$$

Momentum Equation

Assuming that these forces represent the average value over the entire blade length we multiply by the blade length to get the total thrust and torque forces acting on a single blade. The blade length is equal to the radius of the rotor, so the forces are

$$F_{\text{thrust}} = 91\frac{\text{N}}{\text{m}}\times 5\text{ m} = \underline{455\text{ N}}$$

$$F_{\text{torque}} = 25\frac{\text{N}}{\text{m}}\times 5\text{ m} = \underline{125\text{ N}}$$

The torque acting on a single blade can then be estimated to be

$$T = F_{\text{torque}} \times R = 125\text{ N} \times 2.5\text{ m} = \underline{312\text{ N}\cdot\text{m}}$$

The power produced by the three blades on the rotor is approximately

$$P = N \times 2\pi nT = 3 \times 2\pi\left(72\frac{rev}{60\text{ s}}\right) \times 312\text{ N}\cdot\text{m} = \underline{7070\text{ W}}$$

This is only an approximation because of the two-dimensional model used.

Momentum Equation

- **Steady Nonuniform Flow**

- For nonuniform velocity profiles, the momentum flux is:

$$\int_A V^2\, dA = \beta \bar{V}^2 A$$

β is the momentum-correction factor and is:

$$\beta = \frac{\int V^2\, dA}{\bar{V}^2 A}$$

- Through this, the momentum equation for a steady flow with one inlet and one exit:

$$\Sigma \mathbf{F} = \dot{m}(\beta_2 \mathbf{V}_2 - \beta_1 \mathbf{V}_1)$$

Laminar flow with a parabolic profile in a circular pipe has β = 4/3

Momentum Equation

Calculate the momentum correction factor for a parabolic profile **(a)** between parallel plates of width w and **(b)** in a circular pipe of radius R. The parabolic profiles are shown in Figure E4.22.

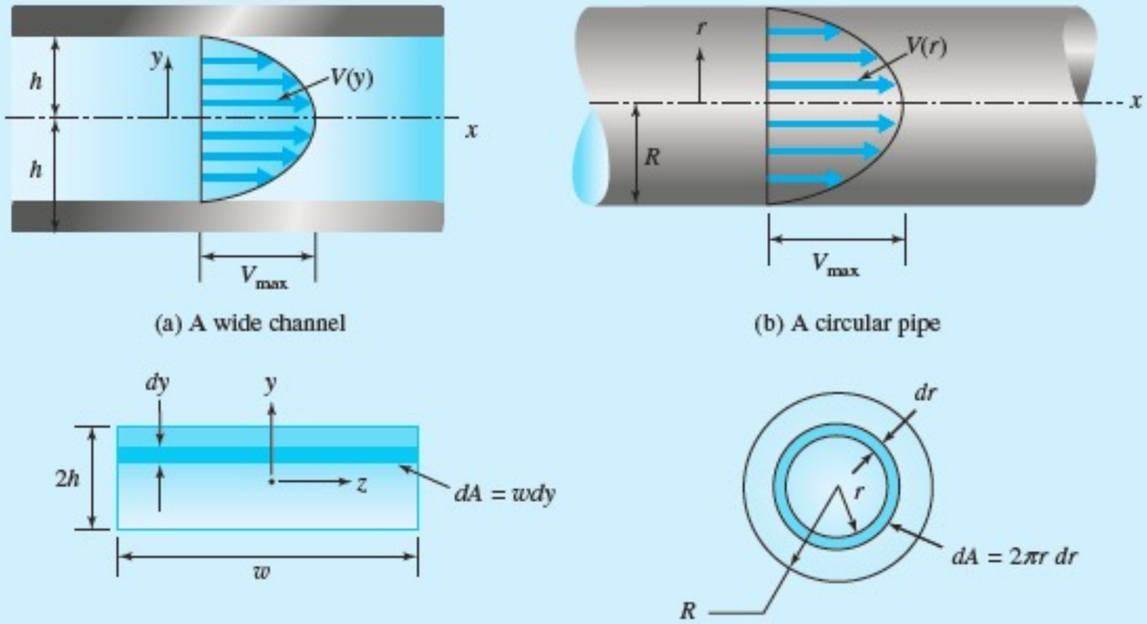

Figure E4.22

Momentum Equation

Solution

(a) A parabolic profile between parallel plates can be expressed as

$$V(y) = V_{\text{max}}\left(1 - \frac{y^2}{h^2}\right)$$

where y is measured from the centerline, the velocity is zero at the walls where $y = \pm h$, and V_{max} is the centerline velocity at $y = 0$. First, let us find the average velocity. It is

$$\bar{V} = \frac{1}{A}\int V\,dA$$
$$= \frac{1}{h\,w}\int_0^h V_{\text{max}}\left(1 - \frac{y^2}{h^2}\right) w\,dy = \frac{V_{\text{max}}}{h}\left(h - \frac{1}{3}h\right) = \frac{2}{3}V_{\text{max}}$$

where we have integrated over the top half of the cross-section. Then

$$\beta = \frac{\int V^2\,dA}{\bar{V}^2 A} = \frac{2}{\frac{4}{9}V_{\text{max}}^2 \times 2h\,w}\int_0^h V_{\text{max}}^2\left(1 - \frac{y^2}{h^2}\right)^2 w\,dy = \underline{1.2}$$

where the factor "2" in the numerator accounts for the bottom half of the channel.

(b) For a circular pipe a parabolic profile can be written as

$$V(r) = V_{\text{max}}\left(1 - \frac{r^2}{R^2}\right)$$

where R is the pipe radius and $V = 0$ at $r = R$. The average velocity is found to be

$$\bar{V} = \frac{1}{A}\int V\,dA = \frac{1}{\pi R^2}\int_0^R V_{\text{max}}\left(1 - \frac{r^2}{R^2}\right)2\pi r\,dr = \frac{1}{2}V_{\text{max}}$$

The momentum correction factor is then

$$\beta = \frac{\int V^2 dA}{\bar{V}^2 A} = \frac{1}{\frac{1}{4}V_{\text{max}}^2 \pi R^2}\int_0^R V_{\text{max}}^2\left(1 - \frac{r^2}{R^2}\right)^2 2\pi r\,dr = \underline{1.33}$$

The correction factors above can be used to express the momentum flux across a cross-sectional area as $\beta\rho A\bar{V}^2$.

Momentum Equation

- **Noninertial Reference Frames**

- Certain applications need a noninertial reference frame to measure velocity.
  - E.g., To study the flow from a rocket or turbine blade, or flow through a dishwasher arm

$$\Sigma \mathbf{F} = \frac{D}{Dt}\int_{\text{sys}} \rho \mathbf{V}\, d\forall + \int_{\text{sys}} \left[\frac{d^2\mathbf{S}}{dt^2} + 2\boldsymbol{\Omega} \times \mathbf{V} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} \right] \rho\, d\forall$$

- The equation above is Newton's Second Law.
  - V: Velocity relative to the noninertial frame.
- Furthermore, the above equation can be rewritten as:

$$\Sigma \mathbf{F} - \mathbf{F}_I = \frac{D}{Dt}\int_{\text{sys}} \rho \mathbf{V}\, d\forall = \frac{d}{dt}\int_{\text{c.v.}} \rho \mathbf{V}\, d\forall + \int_{\text{c.s.}} \rho \mathbf{V}(\mathbf{V}\cdot\hat{\mathbf{n}})\, dA$$

where F_t (the inertial body force) is:

$$\mathbf{F}_I = \int_{\text{sys}} \left[\frac{d^2\mathbf{S}}{dt^2} + 2\boldsymbol{\Omega} \times \mathbf{V} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} \right] \rho\, d\forall$$

Momentum Equation

The rocket shown in Figure E4.23, with an initial mass of 150 kg, burns fuel at the rate of 10 kg/s with a constant exhaust velocity of 700 m/s. What is the initial acceleration of the rocket and the velocity after 1 s? Neglect the drag on the rocket.

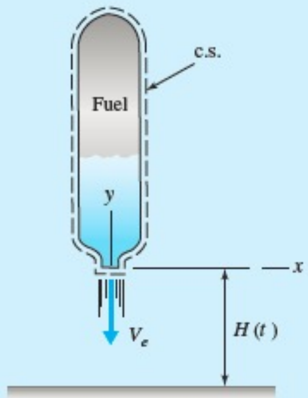

Figure E4.23

Solution

The control volume is sketched and includes the entire rocket. The reference frame attached to the rocket is accelerating upward at d^2H/dt^2. Newton's second law is written as, using z upward,

$$\Sigma F_z - (F_I)_z = \frac{d}{dt}\int_{\text{c.v.}} \rho V_z \, dV\!\!\!/ + \int_{\text{c.s.}} \rho V_z \mathbf{V} \cdot \hat{\mathbf{n}} \, dA$$

$$\therefore \; -W - \frac{d^2 H}{dt^2} m_{\text{c.v.}} = \rho_e(-V_e)V_e A_e$$

where

$$\frac{d}{dt}\int_{\text{c.v.}} \rho V_z \, dV\!\!\!/ \approx 0$$

Momentum Equation

since V_z is the velocity of each mass element $\rho d\mathcal{V}$ relative to the reference frame attached to the control volume, the only vertical force is the weight W, and $m_{c.v.}$ is the mass of the control volume. From continuity we see that

$$m_{c.v.} = 150 - \dot{m}t = 150 - 10t$$
$$\therefore\ W = (150 - 10t) \times 9.81$$

The momentum equation becomes

$$-(150 - 10t)\times 9.81 - \frac{d^2H}{dt^2}(150 - 10t) = -\dot{m}_e V_e = -10 \times 700 = -7000$$

This is written as

$$\frac{d^2H}{dt^2} = \frac{700}{15 - t} - 9.81$$

The initial acceleration is found by letting $t = 0$:

$$\left.\frac{d^2H}{dt^2}\right|_{t=0} = \frac{700}{15} - 9.81 = 36.9\ \text{m/s}^2$$

Integrate the expression for d^2H/dt^2 and obtain

$$\frac{dH}{dt} = -700\ln\ (15 - t) - 9.81t + C$$

The constant $C = 700\ln\ 15$ since $dH/dt = 0$ at $t = 0$. Thus at $t = 1$s the velocity is

$$\frac{dH}{dt} = 700\ln\frac{15}{14} - 9.81 \times 1 = \underline{38.5\ \text{m/s}}$$

Moment-of-Momentum Equation

- Needed to find the line of action of a given force component.
- Needed to analyze flow situations in devices with rotating components (to relate rotational speed to other flow parameters)

Moment-of-Momentum Equation

- The general equation with attached inertial forces is:

$$\Sigma \mathbf{M} - \mathbf{M}_I = \frac{D}{Dt}\int_{\text{sys}} \mathbf{r} \times \mathbf{V} \rho \, d\mathcal{V}$$

where

$$\mathbf{M}_I = \int \mathbf{r} \times \left[\frac{d^2\mathbf{S}}{dt^2} + 2\boldsymbol{\Omega} \times \mathbf{V} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} \right] \rho \, d\mathcal{V}$$

M_I is the inertial moment that accounts for the noninertial reference frame.

Moment-of-Momentum Equation

- When a system-to-control volume transformation is applied, the moment-of-momentum equation becomes:

$$\Sigma \mathbf{M} - \mathbf{M}_I = \frac{d}{dt}\int_{\text{c.v.}} \mathbf{r} \times \mathbf{V}\,\rho\, d\mathcal{V} + \int_{\text{c.s.}} \mathbf{r} \times \mathbf{V}(\mathbf{V}\cdot\hat{\mathbf{n}})\rho\, dA$$

Moment-of-Momentum Equation

A sprinkler has four 50-cm-long arms with nozzles at right angles with the arms and 45° with the ground (Figure E4.20). If the total flow rate is 0.01 m^3/s and a nozzle exit diameter is 12 mm, find the rotational speed of the sprinkler. Neglect friction.

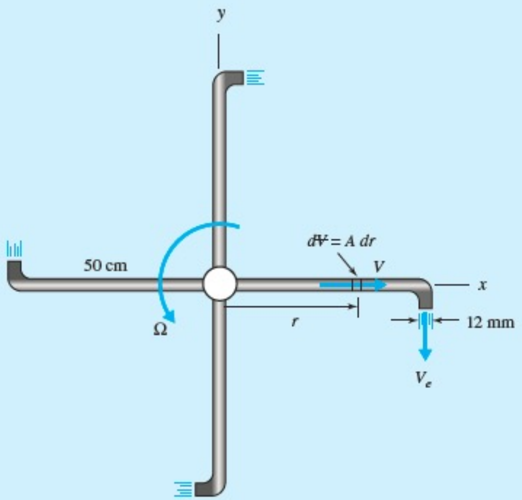

Figure E4.24

Solution

The velocity exiting each nozzle shown is

$$V_e = \frac{Q}{A}$$
$$= \frac{0.01/4}{\pi \times 0.006^2} = 22.1\,\text{m/s}$$

where the factor 4 accounts for the four exit areas. Attach the reference frame to the rotating arms as shown. Then, recognizing that $\mathbf{r} \times [\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})] = 0$ and assuming a stationary sprinkler so that $d^2\mathbf{S}/dt^2 = 0$ and constant angular velocity so that $d\boldsymbol{\Omega}/dt = 0$, we have

$$\mathbf{M}_I = \int_{\text{c.v.}} \mathbf{r} \times (2\boldsymbol{\Omega} \times \mathbf{V})\rho\, d\mathcal{V}$$
$$= 4\int_0^{0.5} r\hat{\mathbf{i}} \times (2\Omega\hat{\mathbf{k}} \times V\hat{\mathbf{i}})\rho A\, dr$$
$$= 8\rho A V \Omega \hat{\mathbf{k}} \int_0^{0.5} r\, dr = \rho A V \Omega \hat{\mathbf{k}}$$

where the small mass of water in the end nozzles is neglected compared to that in the long arms; the factor 4 again accounts for the four arms (each arm would provide the unit vector $\hat{\mathbf{k}}$). Since there are no external moments applied to the sprinkler about the vertical z-axis, $\Sigma M_z = 0$. For the steady flow Eq. 4.7.3 provides

$$\Sigma M_z^{\,0} - (\mathbf{M}_I)_z = \int_{\text{c.s.}} (\mathbf{r} \times \mathbf{V})_z\, \mathbf{V} \cdot \hat{\mathbf{n}}\, \rho\, dA$$
$$-\rho A V \Omega \hat{\mathbf{k}} = 4\int_{A_{exit}} \left[0.5\hat{\mathbf{i}} \times (0.707V_e\hat{\mathbf{k}} - 0.707V_e\hat{\mathbf{j}})\right]_z V_e \rho\, dA$$
$$-VA\Omega = -4 \times 0.5 \times 0.707\, V_e^2 A_e$$
$$\therefore \Omega = 4 \times 0.5 \times 0.707 \times 22.1 = \underline{31.25\ \text{rad/s}}$$

where we have used $AV = A_e V_e$ from continuity considerations.

Moment-of-Momentum Equation

The nozzles of Example 4.24 make an angle of 0° with the ground and 90° with the arms. The water is suddenly turned on at $t = 0$ with the sprinkler motionless. Determine the resulting $\Omega(t)$ if the arm diameter is 24 mm. Neglect friction.

Solution

The reference frame is again attached to the rotating arms, as sketched in Example 4.24. Referring to the control volume integral of Eq. 4.7.3, we observe that $\mathbf{r} \times \mathbf{V} = 0$ since $\mathbf{r}$ is in the same direction as $\mathbf{V}$ along an arm. Thus Eq. 4.7.3, along with Eq. 4.7.2, takes the form

$$\Sigma \cancel{\mathbf{M}}^{0} - 4\int_0^{0.5} r\hat{\mathbf{i}} \times \left[2\Omega\hat{\mathbf{k}} \times V\hat{\mathbf{i}} + \Omega\hat{\mathbf{k}} \times \left(\Omega\hat{\mathbf{k}} \times r\hat{\mathbf{i}}\right) + \frac{d\Omega}{dt}\hat{\mathbf{k}} \times r\hat{\mathbf{i}}\right]\rho A\, dr$$
$$= \frac{d}{dt}\int_{0.5} r\hat{\mathbf{i}} \times V\hat{\mathbf{i}}\rho\, dA + 4\int_{A_{exit}} 0.5\hat{\mathbf{i}} \times V_e\left(-\hat{\mathbf{j}}\right)V_e\rho\, dA$$

Perform the vector operations and divide by 4ρ,

$$-2AV\Omega \int_0^{0.5} r\, dr - \frac{d\Omega}{dt}A\int_0^{0.5} r^2\, dr = -0.5V_e^2 A_e$$

The required integration, using $AV = A_eV_e = 0.01\ \text{m}^3/\text{s}$ and $V_e = 2.21$ m/s gives

$$\frac{d\Omega}{dt} + 132.6\Omega = 5862$$

This linear, first-order differential equation is solved by adding the homogeneous solution (suppress the right-hand side) to the particular solution to obtain

$$\Omega(t) = Ce^{-132.6t} + 44.2$$

Moment-of-Momentum Equation

Using the initial condition $\Omega(0) = 0$, we find that $C = -44.2$. Then

$$\Omega(t) = \underline{44.2(1 - e^{-132.6t})\ \text{rad/s}}$$

Observe that as time becomes large, the angular velocity is limited to 44.2 rad/s. If friction were included, this value would be reduced. If 44.2 is multiplied by 0.707 to account for the 45° angle, we obtain the value of Example 4.24.

Summary

Continuity	*Energy*	*Momentum*
	General Form	
$0 = \frac{d}{dt}\int_{\text{c.v.}} \rho \, d\mathcal{V} + \int_{\text{c.s.}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA$	$-\Sigma \dot{W} = \frac{d}{dt}\int_{\text{c.v.}} \left(\frac{V^2}{2} + gz\right)\rho \, d\mathcal{V} + \int_{\text{c.s.}} \left(\frac{V^2}{2} + \frac{p}{\rho} + gz\right)\rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA + \text{losses}$	$\Sigma \mathbf{F} = \frac{d}{dt}\int_{\text{c.v.}} \rho \mathbf{V} \, d\mathcal{V} + \int_{\text{c.s.}} \rho \mathbf{V}(\mathbf{V} \cdot \hat{\mathbf{n}}) \, dA$
	Steady Flow	
$0 = \int_{\text{c.s.}} \rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA$	$-\Sigma \dot{W} = \int_{\text{c.s.}} \left(\frac{V^2}{2} + \frac{p}{\rho} + gz\right)\rho \mathbf{V} \cdot \hat{\mathbf{n}} \, dA + \text{losses}$	$\Sigma \mathbf{F} = \int_{\text{c.s.}} \rho \mathbf{V}(\mathbf{V} \cdot \hat{\mathbf{n}}) \, dA$

Summary

Steady Nonuniform Form[1]

$$\dot{m} = \rho_1 A_1 \bar{V}_1 = \rho_2 A_2 \bar{V}_2$$

$$\frac{-\Sigma \dot{W}}{\dot{m}g} = \alpha_2 \frac{\bar{V}_2^2}{2g} + \frac{p_2}{\gamma_2} + z_2 - \alpha_1 \frac{\bar{V}_1^2}{2g} - \frac{p_1}{\gamma_1} - z_1 + h_L$$

$$\Sigma F_x = \dot{m}(\beta_2 \bar{V}_{2x} - \beta_1 \bar{V}_{1x})$$
$$\Sigma F_y = \dot{m}(\beta_2 \bar{V}_{2y} - \beta_1 \bar{V}_{1y})$$

Steady Uniform Form[1]

$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

$$-\frac{\Sigma \dot{W}}{\dot{m}g} = \frac{V_2^2}{2g} + \frac{p^2}{\gamma^2} + z_2 - \frac{V_1^2}{2g} - \frac{p_1}{\gamma} - z_1 + h_L$$

$$\Sigma \mathbf{F} = \dot{m}(\mathbf{V}_2 - \mathbf{V}_1)$$

Steady Uniform Incompressible Flow[1]

$$Q = A_1 V_1 = A_2 V_2$$

$$-\frac{\Sigma \dot{W}}{\dot{m}g} = \frac{V_2^2}{2g} + \frac{p^2}{\gamma} + z_2 - \frac{V_1^2}{2g} - \frac{p_1}{\gamma} - z_1 + h_L$$

$$\Sigma \mathbf{F} = \dot{m}(\mathbf{V}_2 - \mathbf{V}_1)$$

$$H_p + \frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = H_T + \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2 + h_L$$

$\dot{m}$ = mass flux

Q = flow rate

$\bar{V}$ = average velocity

$$= \frac{\int V dA}{A}$$

α = kinetic energy correction factor

$$= \frac{\int V^3 dA}{\bar{V}^3 A}$$

β = momentum correction factor

$$= \frac{\int V^2 dA}{\bar{V}^2 A}$$

h_L = head loss

$\Sigma \dot{W} = \dot{W}_S + \dot{W}_{shear} + \dot{W}_I$

H_p = pump head = $\dot{W}_p / \dot{m}g$

H_T = turbine head = $\dot{W}_T / \dot{m}g$

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[1]The control volume has one entrance (section 1) and one exit (section 2).