Dimensional Analysis and Similitude

Introduction

- **Dimensional analysis** is used to keep the required experimental studies to a minimum.
  - Based off **dimensional homogeneity** [all terms in an equation should have the same dimension.]

$$\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2$$

Bernoulli’s equation: Dimension of each term is length

$$\frac{V_1^2}{2gz_1} + \frac{p_1}{\gamma z_1} + 1 = \left(\frac{V_2^2}{2gz_2} + \frac{p_2}{\gamma z_2} + 1\right)\frac{z_2}{z_1}$$

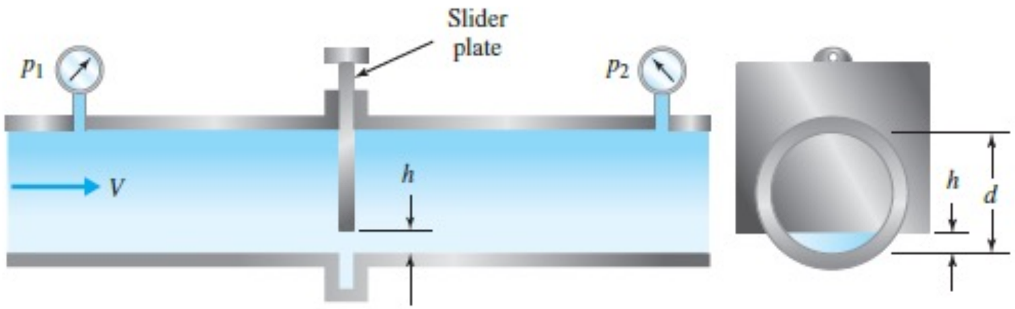
Bernoulli’s equation in this form: Each term is dimensionless

Introduction

- **Similitude** is the study of predicting prototype conditions from model observations.
  - Uses dimensionless parameters obtained in dimensional analysis.
- Two approaches can be used in dimensional analysis:
  - Buckingham π-theorem: Theorem that organizes steps to ensure dimensional homogeneity.
  - Extract dimensionless parameters from the differential equations and boundary conditions.

Dimensional Analysis

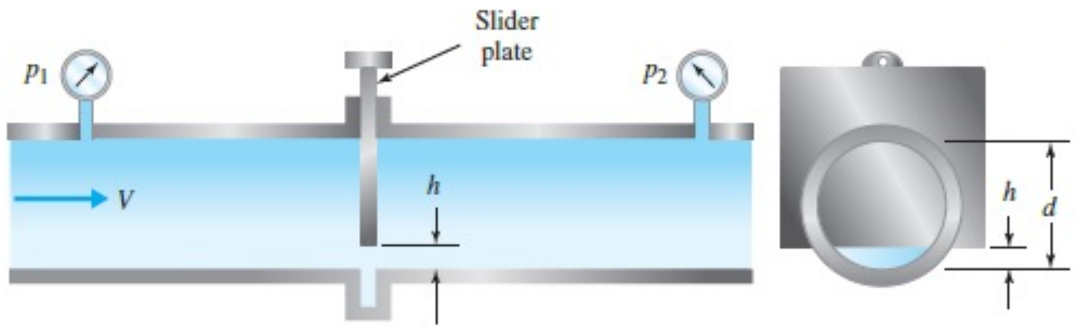
Motivation



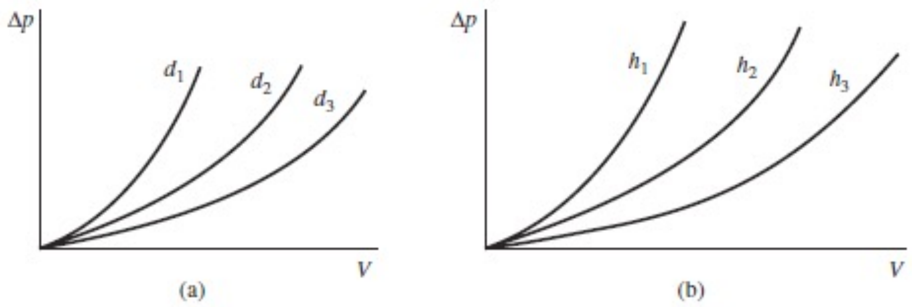
- For pressure drop across a slider valve above:
  - We can assume that it depends on pipe mean velocity V, fluid density ρ, fluid viscosity μ, pipe diameter d, and gap height h [$\Delta p = f(V, \rho, \mu, d, h)$]

Dimensional Analysis

Motivation



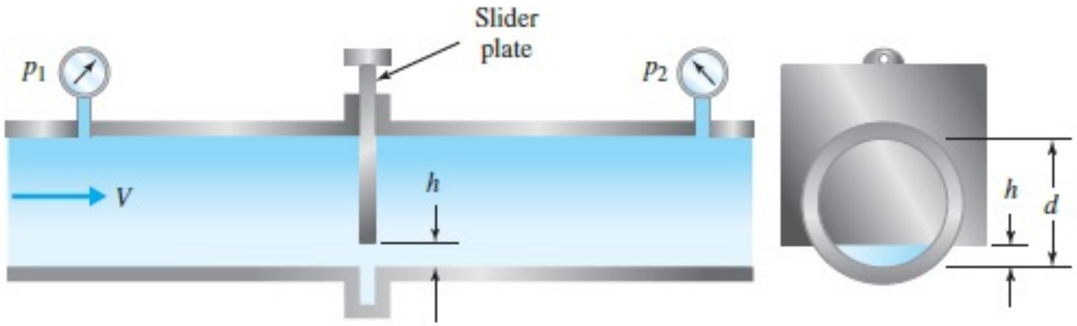
- Could fix all parameters except velocity and find pressure dependence on average velocity.
- Repeat with changing diameter, etc.,



Pressure drop versus velocity curves: (a) ρ, μ, h fixed; (b) ρ, μ, d fixed.

Dimensional Analysis

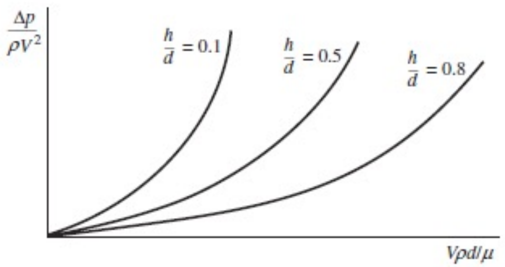
Motivation



$$[\Delta p = f(V, \rho, \mu, d, h)]$$

- The equation could be rewritten in terms of dimensionless parameters as:

$$\frac{\Delta p}{\rho V^2} = f\left(\frac{V\rho d}{\mu}, \frac{h}{d}\right)$$



Dimensionless pressure drop versus dimensionless velocity.

Dimensional Analysis

Review of Dimensions

- All quantities have a combination of dimensions of length, time, mass, and force by Newton's Second Law: $\Sigma \mathbf{F} = m\mathbf{a}$
- In terms of dimensions:

$$F = \frac{ML}{T^2}$$

- To relate thermal effects to the M-L-T system (compressible gas flow), the equation below is used.

$$p = \rho R T \qquad [RT] = [p/\rho] = \frac{F}{L^2} \cdot \frac{L^3}{M} = \frac{ML/T^2}{L^2} \cdot \frac{L^3}{M} = \frac{L^2}{T^2}$$

No additional dimensions

Dimensional Analysis

Symbols and Dimensions of Quantities Used in Fluid Mechanics

Quantity	*Symbol*	*Dimensions*
Length	l	L
Time	t	T
Mass	m	M
Force	F	ML/T^2
Velocity	V	L/T
Acceleration	a	L/T^2
Frequency	ω	T^{-1}
Gravity	g	L/T^2
Area	A	L^2
Flow rate	Q	L^3/T
Mass flux	$\dot{m}$	M/T
Pressure	p	M/LT^2
Stress	τ	M/LT^2
Density	ρ	M/L^3
Specific weight	γ	M/L^2T^2
Viscosity	μ	M/LT
Kinematic viscosity	ν	L^2/T
Work	W	ML^2/T^2
Power, heat flux	$\dot{W}, \dot{Q}$	ML^2/T^3
Surface tension	σ	M/T^2
Bulk modulus	B	M/LT^2

Dimensional Analysis

Buckingham π-Theorem

- In any problem, a dependent variable x_1 is expressed in terms of independent variables, i.e., $x_1 = f(x_2,x_3,x_4,\ldots,x_n)$ *[n: Number of variables]*

π-Terms

- There are (n-m) dimensionless groups of variables, i.e., $\pi_1 = f_1(\pi_2,\pi_3,\ldots,\pi_{n-m})$
  - m: Number of basic dimensions included in the variables.
  - π_1: includes the dependent variable; remaining π-terms include only independent variables.
- For a successful dimensional analysis, a dimension must occur at least twice or not at all.

Dimensional Analysis

Buckingham π-Theorem Procedure

1. Write the functional form of the dependent variable depending on the $(n - 1)$ independent variables. This step requires knowledge of the phenomenon being studied. All variables that effect the dependent variable must be included. These include geometric variables, fluid properties, and external effects that influence the variable being studied. Quantities that have no influence on the dependent variable must not be included. Also, do not include variables that depend on each other; for example, both radius and diameter would not be included. The variables on the right-hand side of Eq. 6.2.7 should be independent.
2. Identify m **repeating variables**, variables that will be combined with each remaining variable to form the π-terms. The repeating variables selected from the independent variables must include all of the basic dimensions, but they must not form a π-term by themselves. An angle cannot be a repeating variable since it is dimensionless and forms a π-term itself.
3. Form the π-terms by combining the repeating variables with each of the remaining variables.
4. Write the functional form of the $(n - m)$ dimensionless π-terms.

Dimensional Analysis

Buckingham π-Theorem Procedure Example

- To combine variables of surface tension σ, velocity V, density ρ, and length l into a π-term.

$$\pi = \sigma^a V^b \rho^c l^d$$

- Need to determine a, b, c, d so that the grouping is dimensionless (Table).

$$M^0 L^0 T^0 = \left(\frac{M}{T^2}\right)^a \left(\frac{L}{T}\right)^b \left(\frac{M}{L^3}\right)^c L^d$$

- Equate the exponents on each of the basic dimensions and solve simultaneously:

$$\begin{aligned} M&: \quad 0 = a + c \\ L&: \quad 0 = b - 3c + d \\ T&: \quad 0 = -2a - b \end{aligned}$$

$$a = -c \qquad b = 2c \qquad d = c$$

Dimensional Analysis

Buckingham π-Theorem Procedure Example

- To combine variables of surface tension σ, velocity V, density ρ, and length l into a π-term.

$$\pi = \sigma^a V^b \rho^c l^d$$

- The π-term becomes:

$$\pi = \left(\frac{\rho l V^2}{\sigma}\right)^c$$

- Can select *c* to be any number other than zero (simplest is 1).

Dimensional Analysis

Buckingham π-Theorem Procedure Example

- NOTE: A dimensionless parameter raised to any power remains dimensionless.

Dimensional Analysis

The drag force F_D on a cylinder of diameter d and length l is to be studied. What functional form relates the dimensionless variables if a fluid with velocity V flows normal to the cylinder?

Solution

First, we must determine the variables that have some influence on the drag force. If we include variables that do not influence the drag force, we would have additional π-terms that experimentation would show to be unimportant; if we do not include a variable that does influence the drag force, experimentation would also reveal that problem. Experience is essential in choosing the correct variables; in this example we will include as influential variables the free stream velocity V, the viscosity μ, the density ρ of the fluid, in addition to the diameter d and the length l of the cylinder, resulting in $n = 6$ variables. This is written as

$$F_D = f(d, l, V, \mu, \rho)$$

The variables are observed to include $m = 3$ dimensions. Refer to Table 6.1 and write

$$[F_D] = \frac{ML}{T^2} \qquad [V] = \frac{L}{T} \qquad [\mu] = \frac{M}{LT} \qquad [d] = L \qquad [l] = L \qquad [\rho] = \frac{M}{L^3}$$

Consequently, we can expect $n - m = 6 - 3 = 3$ π-terms.

We choose repeating variables with the simplest combinations of dimensions such that they do not form a π-term by themselves (we could not include both d and l as repeating variables); the repeating variables are chosen to be d, V, and ρ. These three variables are combined with each of the remaining variables to form the π-terms. Rather than writing equations similar to Eq. 6.2.9 for the π-terms, let us form the π-terms by inspection. When the repeating variables are combined with F_D, we observe that only F_D and ρ have the mass dimension; hence F_D must be divided by ρ. Only F_D and V have the time dimension; hence, F_D must be divided by V^2. Thus F_D divided by ρ has L^4 in the numerator; when divided by V^2 this results in L^2 remaining in the numerator. Hence we must have d^2 in the denominator resulting in

$$\pi_1 = \frac{F_D}{\rho V^2 d^2}$$

Dimensional Analysis

When d, V, and ρ are combined with l there results

$$\pi_2 = \frac{l}{d}$$

The last π-term results from combining μ with d, V, and ρ. The mass dimension disappears if we divide μ by ρ. The time dimension disappears if we divide μ by V. This leaves one length dimension in the numerator; hence d is needed in the denominator resulting in

$$\pi_3 = \frac{\mu}{\rho V d}$$

The dimensionless, functional relationship relating the π-terms is

$$\pi_1 = f_1(\pi_2, \pi_3) \qquad \text{or} \qquad \frac{F_D}{\rho V^2 d^2} = f_1\left(\frac{l}{d}, \frac{\mu}{\rho V d}\right)$$

where $f_1(\ )$ simply means it's different from $f(\ )$.

Rather than the original relationship of six variables we have reduced the relationship to one involving three π-terms, a much simpler expression. To determine the particular form of the functional relationship above, we would actually have to solve the problem; experimentation would be needed if analytical or numerical methods were not available. This is often the case in fluid mechanics.

Note that we could have included several additional variables in our original list, such as gravity g, the angle θ that the velocity makes with the cylinder, and the roughness e of the cylinder surface. To not include variables that are significant, or to include variables that are not significant is a matter of experience. The novice must learn how to identify significant variables; however, even the experienced researcher is often at a loss to correlate certain phenomena; much experimentation is often needed to discover the appropriate parameters.

Dimensional Analysis

The rise of liquid in a capillary tube is to be studied. It is anticipated that the rise h will depend on surface tension σ, tube diameter d, liquid specific weight γ, and angle β of attachment between the liquid and tube. Write the functional form of the dimensionless variables.

Solution

The expression relating the variables is

$$h = f(\sigma, d, \gamma, \beta)$$

The dimensions of the variables are

$$[h] = L \qquad [\gamma] = \frac{M}{L^2T^2} \qquad [\beta] = 1 \text{ (dimensionless)} \qquad [\sigma] = \frac{M}{T^2} \qquad [d] = L$$

By observation we see that M/T^2 occurs as that combination in both σ and γ, hence M and T are not independent dimensions in this problem. There are only two independent groupings of basic dimensions, L and M/T^2. Thus $m = 2$, and we choose σ and d as the repeating variables. When combined with h, the first π-term is

$$\pi_1 = \frac{h}{d}$$

When σ and d are combined with γ, the second π-term is

$$\pi_2 = \frac{\gamma d^2}{\sigma}$$

Finally, since the angle β is dimensionless, it forms a π-term by itself; that is,

$$\pi_3 = \beta$$

Dimensional Analysis

The final functional form relating the π-terms is

$$\pi_1 = f_1(\pi_2, \pi_3) \qquad \text{or} \qquad \frac{h}{d} = f_1\left(\frac{\gamma d^2}{\sigma}, \beta\right)$$

Note: In this example we could not have chosen the angle β as a repeating variable since it already is a dimensionless π-term. Also, we could not have chosen three repeating variables since M and T were not independent.

Also, note that we may have thought that gravity should have been included in the problem. If it had been included above, it would not have appeared in any of the π-terms, indicating that it should not have been included. If density and gravity, rather than specific weight, had been included, the relationship above would have resulted since $\gamma = \rho g$; this, by the way, would have avoided the necessity of observing that M/T^2 was a dimensional grouping.

A final note regarding the functional form of the π-terms: The relationship above could equally have been written as

$$\frac{h}{d} = f_1\left(\frac{\sigma}{\gamma d^2}, \beta\right)$$

Also, occasionally a different set of repeating variables could be selected. This simply expresses the final functional equation in a different but equivalent form. Actually, a second form can be shown to be a combination of the π-terms from an initial form.

Dimensional Analysis

Common Dimensionless Parameters

- For a relationship between pressure drop Δp, characteristic length l, characteristic velocity V, density ρ, viscosity μ, gravity g, surface tension σ, speed of sound c, and angular frequency ω, i.e., $\Delta p = f(l, V, \rho, \mu, g, c, \omega, \sigma)$

- Using the π-theorem, with l, V, and ρ as repeating variables gives:

$$\frac{\Delta p}{\rho V^2} = f_1\left(\frac{V\rho l}{\mu}, \frac{V^2}{lg}, \frac{V}{c}, \frac{l\omega}{V}, \frac{V^2 \rho l}{\sigma}\right)$$

Dimensional Analysis

Common Dimensionless Parameters

- Each of the π-terms in the equation appears in many fluid flow situations.

$$\frac{\Delta p}{\rho V^2} = f_1\left(\frac{V\rho l}{\mu}, \frac{V^2}{lg}, \frac{V}{c}, \frac{l\omega}{V}, \frac{V^2 \rho l}{\sigma}\right)$$

Euler number, Eu $= \dfrac{\Delta p}{\rho V^2}$

Reynolds number, Re $= \dfrac{V\rho l}{\mu}$

Froude number[2], Fr $= \dfrac{V}{\sqrt{lg}}$

Mach number, M $= \dfrac{V}{c}$

Strouhal number[2], St $= \dfrac{l\omega}{V}$

Weber number[2], We $= \dfrac{V^2 l\rho}{\sigma}$

Dimensional Analysis

Common Dimensionless Parameters

- Each dimensionless number can be written as a ratio of two forces.

$$
\begin{aligned}
F_P &= \text{pressure force} = \Delta p A \sim \Delta p l^2 \\
F_I &= \text{inertial force} = mV\frac{dV}{ds} \sim \rho l^3 V \frac{V}{l} = \rho l^2 V^2 \\
F_\mu &= \text{viscous force} = \tau A = \mu \frac{du}{dy} A \sim \mu \frac{V}{l} l^2 = \mu l V \\
F_g &= \text{gravity force} = mg \sim \rho l^3 g \\
F_B &= \text{compressibility force} = BA \sim \rho \frac{dp}{d\rho} l^2 = \rho c^2 l^2 \\
F_\omega &= \text{centrifugal force} = mr\omega^2 \sim \rho l^3 l \omega^2 = \rho l^4 \omega^2 \\
F_\sigma &= \text{surface tension force} = \sigma l
\end{aligned}
$$

$$
\begin{aligned}
\text{Eu} &\propto \frac{\text{pressure force}}{\text{inertial force}} \\
\text{Re} &\propto \frac{\text{inertial force}}{\text{viscous force}} \\
\text{Fr} &\propto \frac{\text{inertial force}}{\text{gravity force}} \\
\text{M} &\propto \frac{\text{inertial force}}{\text{compressibility force}} \\
\text{St} &\propto \frac{\text{centrifugal force}}{\text{inertial force}} \\
\text{We} &\propto \frac{\text{inertial force}}{\text{surface tension force}}
\end{aligned}
$$

Dimensional Analysis

Common Dimensionless Parameters

Common Dimensionless Parameters in Fluid Mechanics

Parameter	*Expression*	*Flow situations where parameter is important*
Euler number	$\dfrac{\Delta p}{\rho V^2}$	Flows in which pressure drop is significant: most flow situations
Reynolds number	$\dfrac{\rho l V}{\mu}$	Flows that are influenced by viscous effects: internal flows, boundary layer flows
Froude number	$\dfrac{V}{\sqrt{lg}}$	Flows that are influenced by gravity: primarily free surface flows
Mach number	$\dfrac{V}{c}$	Compressibility is important in these flows, usually if $V > 0.3\,c$
Strouhal number	$\dfrac{l\omega}{V}$	Flow with an unsteady component that repeats itself periodically
Weber number	$\dfrac{V^2 l \rho}{\sigma}$	Surface tension influences the flow; flow with an interface may be such a flow

Similitude

General Information

- Study of predicting prototype conditions from model observations.
- If a model study has to be performed:
  - Need a quantity measured on the model (subscript *m*) to predict an associated quantity on the prototype (subscript *p*).
  - This needs **dynamic similarity between the model and prototype.**
  - Forces which act on corresponding masses in the model flow and prototype flow are in the same ratio throughout the entire flows.

Similitude

General Information

- If inertial forces, pressure forces, viscous forces, and gravity forces are present:

$$\frac{(F_I)_m}{(F_I)_p} = \frac{(F_P)_m}{(F_P)_p} = \frac{(F_\mu)_m}{(F_\mu)_p} = \frac{(F_g)_m}{(F_g)_p} = \text{const.}$$

Due to dynamic similarity at corresponding points in the flow fields.

Rearrange

$$\left(\frac{F_I}{F_P}\right)_m = \left(\frac{F_I}{F_P}\right)_p \qquad \left(\frac{F_I}{F_\mu}\right)_m = \left(\frac{F_I}{F_\mu}\right)_p \qquad \left(\frac{F_I}{F_g}\right)_m = \left(\frac{F_I}{F_g}\right)_p$$

$$\mathrm{Eu}_m = \mathrm{Eu}_p \qquad \mathrm{Re}_m = \mathrm{Re}_p \qquad \mathrm{Fr}_m = \mathrm{Fr}_p$$

Similitude

General Information

- If only the forces above (previous slide) are present: $F_I = f(F_P, F_\mu, F_g)$

- Dimensional analysis lets the equation be written in terms of force ratios, as there is only one main dimension.

$$\mathrm{Eu} = f(\mathrm{Re}, \mathrm{Fr})$$

- If the Reynolds and Froude numbers are the same on the model and prototype, the Euler number should be the same.

- Guarantee dynamic similarity between model and prototype by equating the Reynolds number and Froude number of the model to that of the prototype.

Similitude

General Information

- If compressibility forces are included, Mach number would be included.
- The inertial force ratio would be:

$$\frac{(F_I)_m}{(F_I)_p} = \frac{a_m m_m}{a_p m_p} = \text{const.}$$

If the mass ratio is a constant, then the acceleration ratio is a constant

Hence:

$$\frac{a_m}{a_p} = \frac{V_m^2/l_m}{V_p^2/l_p} = \text{const.}$$

Similitude

General Information

- **Kinematic Similarity**: Velocity ratio is a constant between all corresponding points in the flow fields.
    - **Streamline pattern around the model is the same as that around the prototype except for a scale factor.**
- **Geometric Similarity**: Length ratio is a constant between all corresponding points in the flow fields.
    - **Model has the same shape as the prototype.**

Similitude

General Information

For complete similarity between the model and prototype

- **Geometric similarity must be satisfied.**
- **Mass ratio of corresponding fluid elements is a constant.**
- **Dimensionless parameters (below) should be equal.**

$$\text{Euler number, Eu} = \frac{\Delta p}{\rho V^2}$$

$$\text{Reynolds number, Re} = \frac{V\rho l}{\mu}$$

$$\text{Froude number}^2,\ \text{Fr} = \frac{V}{\sqrt{lg}}$$

$$\text{Mach number, M} = \frac{V}{c}$$

$$\text{Strouhal number}^2,\ \text{St} = \frac{l\omega}{V}$$

$$\text{Weber number}^2,\ \text{We} = \frac{V^2 l\rho}{\sigma}$$

Similitude

General Information

- Can now *predict* quantities of interest on a prototype from measurements on a model.

Drag forces, F_D	$\frac{(F_D)_m}{(F_D)_p} = \frac{(F_I)_m}{(F_I)_p} = \frac{\rho_m V_m^2 l_m^2}{\rho_p V_p^2 l_p^2}$	Equate ratio of drag forces to ratio of inertial forces.
Power input, $\dot{W}$	$\frac{\dot{W}_m}{\dot{W}_p} = \frac{(F_I)_m V_m}{(F_I)_p V_p} = \frac{\rho_m V_m^2 l_m^2 V_m}{\rho_p V_p^2 l_p^2 V_p}$	Power is force times velocity.

- Can predict a prototype quantity if we select the model fluid, the scale fluid, and the dimensionless number.

Similitude

Confined Flows

- A confined flow is a flow that has no free surface (liquid-gas surface) or interface (two different liquids).

- Can only move within a specific region (external flows around objects, or internal flows in pipes).

- Isn't influenced by gravity or surface tension.
- Dominant effect is viscosity in incompressible confined flows.
- Relevant flows are pressure, inertial, and viscous forces.
    - Dynamic similarity is obtained if the ratios between the model and the prototype are the same.
- **Hence, only the Reynolds number is the dominant dimensionless parameter.**
    - **If compressibility effects are significant, Mach number would become important.**

Similitude

A test is to be performed on a proposed design for a large pump that is to deliver 1.5 m³/s of water from a 400-mm-diameter impeller with a pressure rise of 400 kPa. A model with an 80-mm-diameter impeller is to be used. What flow rate should be used and what pressure rise is to be expected? The model fluid is water at the same temperature as the water in the prototype.

Solution

For similarity to exist in this confined incompressible flow problem, the Reynolds numbers must be equal; that is,

$$\mathrm{Re}_m = \mathrm{Re}_p$$

$$\frac{V_m d_m}{\nu_m} = \frac{V_p d_p}{\nu_p}$$

Recognizing that $\nu_m = \nu_p$ if the temperatures are equal, we see that

$$\frac{V_m}{V_p} = \frac{d_p}{d_m} = \frac{0.4}{0.08} = 5$$

The ratio of flow rates is found recognizing that $Q = VA$:

$$\frac{Q_m}{Q_p} = \frac{V_m d_m^2}{V_p d_p^2} = 5 \times \left(\frac{1}{5}\right)^2 = \frac{1}{5}$$

Thus we find that

$$Q_m = \frac{Q_p}{5} = \frac{1.5}{5} = \underline{0.3\ \mathrm{m^3/s}}$$

The dimensionless pressure rise is found using the Euler number:

$$\left(\frac{\Delta p}{\rho V^2}\right)_m = \left(\frac{\Delta p}{\rho V^2}\right)_p$$

Hence the pressure rise for the model is

$$\Delta p_m = \Delta p_p \frac{\rho_m}{\rho_p} \frac{V_m^2}{V_p^2} = 400 \times 1 \times 5^2 = \underline{10\,000\ \mathrm{kPa}}$$

Note that in this example we see that the velocity in the model is equal to the velocity in the prototype multiplied by the length ratio, and the pressure rise in the model is equal to the pressure rise in the prototype multiplied by the length ratio squared. If the length ratio were very large, it is obvious that to maintain Reynolds number equivalence would be quite difficult, indeed. This observation is discussed in more detail in Section 6.3.4.

Similitude

Free-Surface Flows

- A free-surface flow is a flow where part of the boundary involves a pressure boundary condition.
  - E.g., Flows in channels, flows with two fluids separated by an interface, etc.

- Location and velocity of the free surface are unknown.
- Pressure is the same on either side of the interface (unless there is significant surface tension).
- Gravity controls the location and motion of the free surface.
- Viscous effects are significant

- **Requires the Froude number.**

Similitude

A 1:20 scale model of a surface vessel is used to test the influence of a proposed design on the wave drag. A wave drag of 27.6 N is measured at a model speed of 2.44 m/s. What speed does this correspond to on the prototype, and what wave drag is predicted for the prototype? Neglect viscous effects, and assume the same fluid for model and prototype.

Solution

The Froude number must be equated for both model and prototype. Thus

$$\mathrm{Fr}_m = \mathrm{Fr}_p \qquad \frac{V_m}{\sqrt{l_m g}} = \frac{V_p}{\sqrt{l_p g}}$$

This yields, recognizing that g does not vary significantly on the surface of the earth,

$$V_p = V_m \left(\frac{l_p}{l_m}\right)^{1/2} = 2.44\sqrt{20} = \underline{10.9 \text{ m/s}}$$

To find the wave drag on the prototype, we equate the drag ratio to the inertia force ratio:

$$\frac{(F_D)_m}{(F_D)_p} = \frac{\rho_m V_m^2 l_m^2}{\rho_p V_p^2 l_p^2}$$

This allows us to calculate the wave drag on the prototype as, using $\rho_p = \rho_m$,

$$\begin{aligned}(F_D)_p &= (F_D)_m \frac{\rho_p V_p^2 l_p^2}{\rho_m V_m^2 l_m^2} \\ &= 27.6 \times \frac{10.9^2}{2.44^2} \times 20^2 = \underline{220 \text{ kN}}\end{aligned}$$

Note: We could have used the gravity force ratio rather than the inertial force ratio, but we could not have used the viscous force ratio since viscous forces were assumed negligible.

Similitude

A 1:10 scale model of an automobile is used to measure the drag on a proposed design. It is to simulate a prototype speed of 90 km/h. What speed should be used in the wind tunnel if Reynolds numbers are equated? For this condition, what is the ratio of drag forces?

Solution

The same fluid exists on model and prototype; thus, equating the Reynolds numbers results in

$$\frac{V_m l_m}{\nu_m} = \frac{V_p l_p}{\nu_p} \qquad \therefore V_m = V_p \frac{l_p}{l_m}$$

$$= 90 \times 10 = \underline{900 \text{ km/h}}$$

This speed would, of course, introduce compressibility effects, effects that do not exist in the prototype. Hence the proposed model study would be inappropriate.

If we did use this velocity in the model, the drag force ratio would be

$$\frac{(F_D)_p}{(F_D)_m} = \frac{\rho_p V_p^2 l_p^2}{\rho_m V_m^2 l_m^2} \qquad \therefore \frac{(F_D)_p}{(F_D)_m} = 1$$

Thus we see that the drag force on the model is the same as the drag force on the prototype if the same fluids are used when we equate Reynolds numbers.

Similitude

In Example 6.5, if the Reynolds numbers were equated, the velocity in the model study was observed to be in the compressible flow regime (i.e., $M > 0.3$ or $V_m > 360$ km/h). To conduct an acceptable model study, could we use a velocity of 90 km/h on a model with a characteristic length of 10 cm? Assume that the drag coefficient ($C_D = F_D / \frac{1}{2}\rho V^2 A$, where A is the projected area) is independent of Re for Re $> 10^5$. If so, what drag force on the prototype would correspond to a drag force of 1.2 N measured on the model?

Solution

The proposed model study in a wind tunnel is to be conducted with $V_m = 90$ km/h and $l_m = 0.1$ m. Using $\nu = 1.6 \times 10^{-5}$ m²/s, the Reynolds number is

$$\mathrm{Re}_m = \frac{V_m l_m}{\nu_m} = \frac{(90 \times 1000/3600) \times 0.1}{1.6 \times 10^{-5}} = 1.56 \times 10^5$$

This Reynolds number is greater than 10^5, so we will assume that similarity exists between model and prototype. The velocity of 90 km/h is sufficiently high.

The drag force on the prototype traveling at 90 km/h corresponding to 1.2 N on the model is found from

$$\frac{(F_D)_p}{(F_D)_m} = \frac{\rho_p V_p^2 l_p^2}{\rho_m V_m^2 l_m^2} \qquad \therefore (F_D)_p = (F_D)_m \cancelto{1}{\frac{\rho_p}{\rho_m}} \cancelto{1}{\frac{V_p^2}{V_m^2}} \frac{l_p^2}{l_m^2} = 1.2 \times 10^2 = \underline{120\ \mathrm{N}}$$

Note that in this example we have assumed that the drag coefficient is independent of Re for Re $> 10^5$. If the drag coefficient continued to vary above Re $= 10^5$ (this would be evident from experimental data), the foregoing analysis would have to be modified accordingly.

Similitude

Compressible Flows

- For most compressible flows, the Reynolds number is very large (not significant).
  - Mach number is the primary dimensionless parameter for model studies.

$$M_m = M_p \qquad \text{or} \qquad \frac{V_m}{c_m} = \frac{V_p}{c_p}$$

- If the study is carried in a wind tunnel (with a prototype fluid of air), $c_m = c_p$.
  - Assume the temperature is the same in both flows.

- In this case, the velocity in the model study is equal to the velocity associated with the prototype.

Similitude

The pressure rise from free stream to the nose of a fusilage section of an aircraft is measured in a wind tunnel at 20°C to be 34 kPa with a wind-tunnel airspeed of 900 km/h. If the test is to simulate flight at an elevation of 12 km, what is the prototype velocity and the expected nose pressure rise?

Solution

To find the prototype velocity corresponding to a wind-tunnel airspeed of 900 km/h, we equate the Mach numbers

$$M_m = M_p \qquad \text{or} \qquad \frac{V_m}{\sqrt{kRT_m}} = \frac{V_p}{\sqrt{kRT_p}}$$

Thus

$$V_p = V_m \left(\frac{kRT_p}{kRT_m} \right)^{1/2} = 900 \left(\frac{216.7}{293} \right)^{1/2} = \underline{774 \text{ km/h}}$$

The pressure at the nose of the prototype fusilage is found using the Euler number as follows:

$$\frac{\Delta p_m}{\rho_m V_m^2} = \frac{\Delta p_p}{\rho_p V_p^2}$$

$$\therefore \Delta p_p = \Delta p_m \frac{\rho_p}{\rho_m} \frac{V_p^2}{V_m^2}$$

$$= 34 \times \frac{0.3119}{1.225} \times \frac{774^2}{900^2} = \underline{6.4 \text{ kPa}}$$

The densities and temperature T_p were found in Appendix B.

Similitude

Periodic Flows

- There are regions of flows in which periodic motions occur.
  - E.g., When fluid flows past a cylindrical object.

- **For these flows, we need to equate Strouhal numbers to model the periodic motion.**

$$\frac{V_m}{\omega_m l_m} = \frac{V_p}{\omega_p l_p}$$

- Additional dimensionless parameters that may be equated.
  - **In viscous flows → Reynolds number**
  - **In free-surface flows → Froude number**
  - **In compressible flows → Mach number**

Similitude

A large wind turbine, designed to operate at 50 km/h, is to be tested in a laboratory by constructing a 1:15 scale model. What airspeed should be used in the wind tunnel, what angular velocity should be used to simulate a prototype angular speed of 5 rpm, and what power output is expected from the model if the prototype output is designed to be 500 kW?

Solution

The speed in the wind tunnel can be any speed above that needed to provide a sufficiently large Reynolds number. Let us select the same speed with which the prototype is to operate, namely, 50 km/h, and calculate the minimum characteristic length that a Reynolds number of 10^5 would demand; this gives

$$\mathrm{Re} = \frac{Vl}{\nu} \qquad 10^5 = \frac{(50 \times 1000/3600) \times l}{1.6 \times 10^{-5}} \qquad \therefore l = 0.12 \text{ m}$$

Obviously, in a reasonably large wind tunnel we can maintain a characteristic length (e.g., the blade length) that large.

The angular velocity is found by equating the Strouhal numbers. There results

$$\frac{V_m}{\omega_m l_m} = \frac{V_p}{\omega_p l_p} \qquad \therefore \omega_m = \omega_p \cancelto{1}{\frac{V_m}{V_p}} \frac{l_p}{l_m} = 5 \times 1 \times 15 = \underline{75 \text{ rpm}}$$

assuming that the wind velocities are equal.

The power is found by observing that power is force times velocity:

$$\frac{\dot{W}_m}{\dot{W}_p} = \frac{\rho_m V_m^3 l_m^2}{\rho_p V_p^3 l_p^2}$$

or

$$\dot{W}_m = \dot{W}_p \cancelto{1}{\frac{\rho_m}{\rho_p}} \cancelto{1}{\frac{V_m^3}{V_p^3}} \frac{l_m^2}{l_p^2} = 500 \times \left(\frac{1}{15}\right)^2 = \underline{2.22 \text{ kW}}$$

Normalized Differential Equations

- Usually when using differential equations to describe the laminar/turbulent, steady/unsteady, compressible/incompressible, confined/free-surface flows, etc.→ express in dimensionless/normalized form.

- In vector form, the continuity equation and the Navier-Stokes equations are:

$$\nabla \cdot \mathbf{V} = 0$$

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p - \rho g \nabla h + \mu \nabla^2 \mathbf{V}$$

- In terms of **kinetic pressure**, the Navier-Stokes becomes:

$$\boxed{\rho \frac{D\mathbf{V}}{Dt} = -\nabla p_k + \mu \nabla^2 \mathbf{V}}$$

Kinetic Pressure: Pressure resulting from fluid motion alone

Normalized Differential Equations

- To normalize the differential equations, we need to select characteristic quantities:

$$u^* = \frac{u}{V} \qquad v^* = \frac{v}{V} \qquad w^* = \frac{w}{v} \qquad x^* = \frac{x}{l} \qquad y^* = \frac{y}{l} \qquad z^* = \frac{z}{l}$$

Asterisks: Dimensionless quantities

$$p^* = \frac{p}{\rho V^2} \qquad t^* = \frac{t}{l/V}$$

- Hence:

$$\frac{D^*\mathbf{V}^*}{Dt^*} = -\boldsymbol{\nabla}^* p_k^* + \frac{\mu}{\rho V l} \nabla^{*2}\mathbf{V}^*$$

$$\boldsymbol{\nabla}^* \cdot \mathbf{V}^* = 0$$

$$\begin{aligned} \mathbf{V}^* &= u^*\hat{\mathbf{i}} + v^*\hat{\mathbf{j}} + w^*\hat{\mathbf{k}} \\ &= \frac{u}{V}\hat{\mathbf{i}} + \frac{v}{V}\hat{\mathbf{j}} + \frac{w}{V}\hat{\mathbf{k}} = \frac{\mathbf{V}}{V} \end{aligned}$$

$$\begin{aligned} \boldsymbol{\nabla}^* &= \frac{\partial}{\partial x^*}\hat{\mathbf{i}} + \frac{\partial}{\partial y^*}\hat{\mathbf{j}} + \frac{\partial}{\partial z^*}\hat{\mathbf{k}} \\ &= l\frac{\partial}{\partial x}\hat{\mathbf{i}} + l\frac{\partial}{\partial y}\hat{\mathbf{j}} + l\frac{\partial}{\partial z}\hat{\mathbf{k}} = l\,\boldsymbol{\nabla} \end{aligned}$$

Normalized Differential Equations

- The **Reynolds number** as a parameter in the normalized Navier-Stokes equation is:

$$\mathrm{Re} = \frac{\rho V l}{\mu}$$

- If a portion of the boundary is oscillating, the **Strouhal number** should be introduced.
  - Velocity of the fluid is the same as that of the rotating part.

$$v^* = \mathrm{St}\, r^*$$

$$\mathrm{St} = \frac{\omega l}{V}$$

Normalized Differential Equations

- A boundary condition that introduces another parameter is the free-surface, which introduces the **Froude number**.

$$\mathrm{Fr} = \frac{V}{\sqrt{lg}}$$

- The **Weber number** is introduced for a boundary condition that involves surface tension.

$$\mathrm{We} = \frac{V^2 l \rho}{\sigma}$$

Summary

- The most common flow parameters that use *l* as a characteristic length are:

$$\mathrm{Re} = \frac{V\rho l}{\mu}, \quad \mathrm{Fr} = \frac{V}{\sqrt{lg}}, \quad \mathrm{M} = \frac{V}{c}, \quad \mathrm{Eu} = \frac{\Delta p}{\rho V^2}, \quad \mathrm{St} = \frac{l\omega}{V}, \quad \mathrm{We} = \frac{V^2 l\rho}{\sigma}$$

Confined flows:	$\mathrm{Re} = \frac{V\rho l}{\mu}$
Free-surface flows:	$\mathrm{Fr} = \frac{V}{\sqrt{lg}}$
High-Reynolds-number flows:	$\mathrm{Re} > (\mathrm{Re})_{\mathrm{minimum}}$
Compressible flows:	$\mathrm{M} = \frac{V}{c}$
Periodic flows:	$\mathrm{St} = \frac{V}{l\omega}$

Summary

- The Navier-Stokes equation in terms of dimensionless variables is:

$$\frac{D\mathbf{V}}{Dt} = -\boldsymbol{\nabla} p_k + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{V}$$

  - p_k is the kinetic pressure.
  - All variables are dimensionless since the Reynolds number appears in the equation.