Chapter 1

The Differential Forms of the Fundamental Laws

5.1 Introduction

- Two primary methods in deriving the differential forms of fundamental laws:
  - Gauss's Theorem: Allows area integrals of the equations (of the previous chapter) to be transformed to volume integrals (and then set to zero).
    - Valid over any arbitrary control volume.
  - (Used in this book) Identify an infinitesimal element in space and apply the basic laws to those elements.
    - Easier math/computation.

- Conservation of mass (to an infinitesimal element) → Differential continuity equation (density and velocity fields)
- Newton's Second law → Navier-Stokes equations (velocity, pressure, and density field relationship)
- First Law of thermodynamics → Differential energy equation (temperature field to velocity, density, and pressure fields)

5.1 Introduction

- Most problems are assumed to be isothermal, incompressible flows in which the temperature field doesn't play a role.

- Initial Conditions: Conditions (independent variable) that depend on time.
- Boundary Conditions: Conditions (independent variable) that depend on a spatial coordinate.
  - No-slip conditions for a viscous flow. Velocity of the fluid at the wall equals the velocity at the wall (usually stationary).
  - Normal component of velocity in an inviscid flow (negligible viscous effects).
  - Pressure in a flow with a free-surface.
  - Temperature of the boundary (temperature gradient at the boundary). For a constant boundary temperature, the temperature of the fluid next to the boundary equals the boundary temperature.

5.2 Differential Continuity Equation

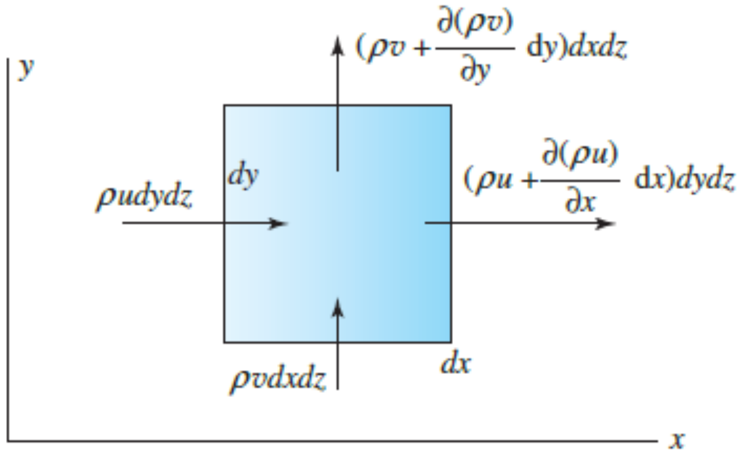


Figure 5.1 An infinitesimal control volume using rectangular coordinates.

- Net mass flux entering the element equals the rate of change of the mass of the element.

$$\dot{m}_{\text{in}} - \dot{m}_{\text{out}} = \frac{\partial}{\partial t} m_{\text{element}}$$

- For a 2-dimensional flow (xy plane), using the diagram:

$$\rho udydz + \rho vdxdz - \left(\rho u + \frac{\partial(\rho u)}{\partial x} dx\right)dydz - \left(\rho v + \frac{\partial(\rho v)}{\partial y} dy\right)dxdz = \frac{\partial}{\partial t}(\rho dxdydz)$$

5.2 Differential Continuity Equation

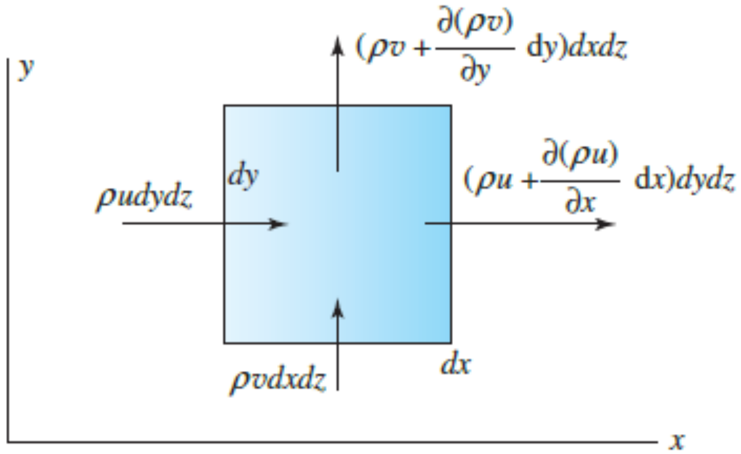


Figure 5.1 An infinitesimal control volume using rectangular coordinates.

- After rearranging the equation on the previous slide, a general form of the **differential continuity equation** is obtained.
  - Rectangular coordinates.

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

- A gradient operator (del, ∇) is introduced as: $\boldsymbol{\nabla} = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}$

5.2 Differential Continuity Equation

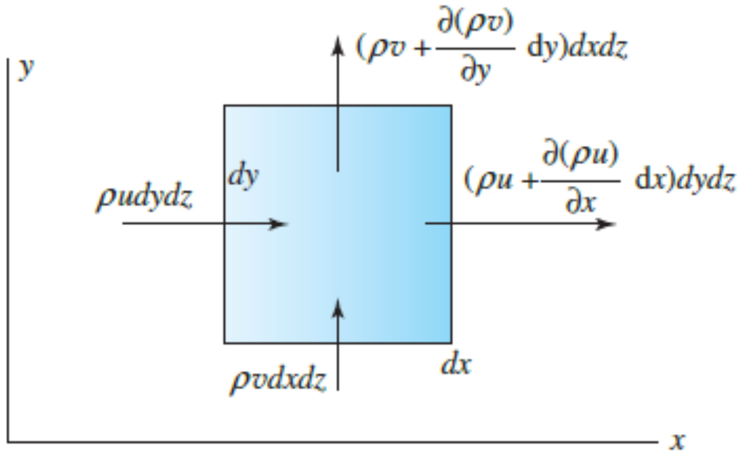


Figure 5.1 An infinitesimal control volume using rectangular coordinates.

- A different form of the continuity equation is:

$$\frac{D\rho}{Dt} + \rho\, \nabla \cdot \mathbf{V} = 0$$

- $V = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}} + w\widehat{\boldsymbol{k}}$
- This dot-product $\nabla \cdot V$ is called the **velocity divergence**.

5.2 Differential Continuity Equation

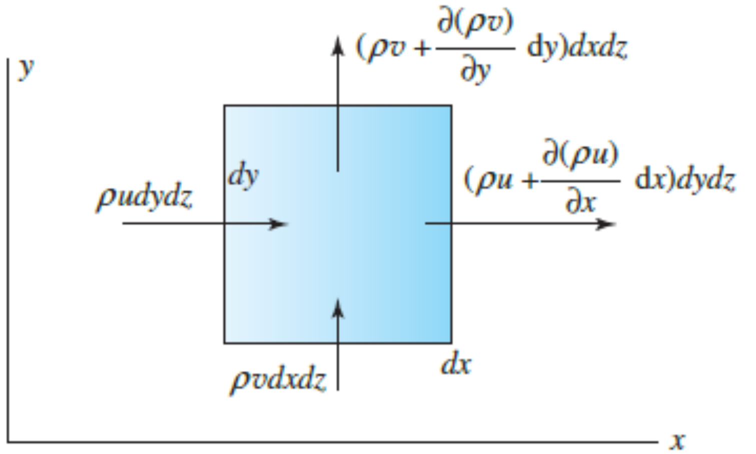


Figure 5.1 An infinitesimal control volume using rectangular coordinates.

Incompressible Flow

- Does not demand that ρ is constant.
- Instead, the density of the fluid particle does not change as it travels along, i.e.,

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0$$

- Slightly different from assumption of *constant density*:
  - Which means that each term in the above equation has to be zero.

5.2 Differential Continuity Equation

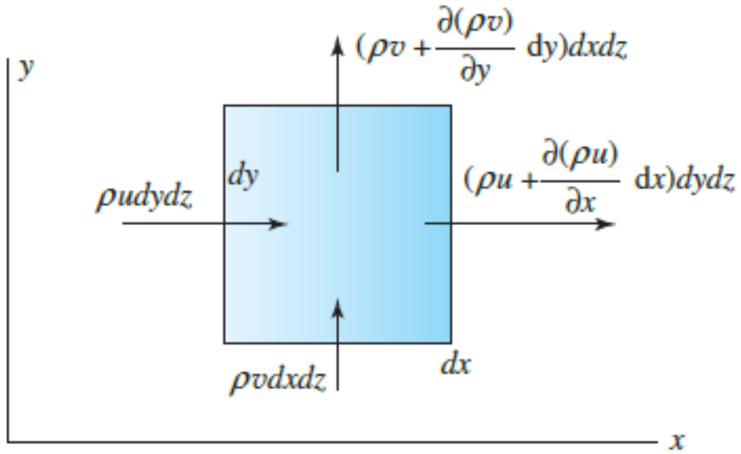


Figure 5.1 An infinitesimal control volume using rectangular coordinates.

Incompressible Flow

- Incompressible flows with density gradients are also referred to as *stratified flows* or *nonhomogeneous flows.*
- The continuity equation for this flow is either:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \nabla \cdot \mathbf{V} = 0$$

  - The divergence of the velocity vector is zero for an incompressible flow (even if the flow is unsteady).

5.2 Differential Continuity Equation

The x-component velocity is given by $u(x, y) = Ay^2$ in an incompressible plane flow. Determine $v(x, y)$ if $v(x, 0) = 0$, as would be the case in flow between parallel plates.

Solution

The differential continuity equation for an incompressible, plane flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

since in a plane flow the two velocity components depend only on x and y. Using the given $u(x, y)$ we find that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(Ay^2) + \frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial v}{\partial y} = 0$$

Since this is a partial differential equation, its solution is

$$v(x, y) = f(x)$$

But $v(x, 0) = 0$ requiring that $f(x) = 0$. Consequently,

$$\underline{v(x, y) = 0}$$

is the y-component velocity demanded by the conservation of mass. In order for $v(x, y)$ to be nonzero, $u(x, y)$ would have to vary with x or $v(x, 0)$ would have to be nonzero.

5.2 Differential Continuity Equation

Air flows in a pipe and the velocity at three neighboring points A, B, and C, 10 cm apart, is measured to be 83, 86, and 88 m/s, respectively, as shown in Figure E5.2. The temperature and pressure are 10°C and 345 kPa, respectively, at point B. Approximate $d\rho/dx$ at that point, assuming steady, uniform flow.

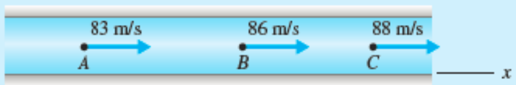


Figure E5.2

Solution

The continuity equation (5.2.5) for this steady $\left(\frac{\partial}{\partial t} = 0\right)$, uniform $\left(\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0\right)$ flow reduces to

$$u\frac{d\rho}{dx} + \rho\frac{du}{dx} = 0$$

We used ordinary derivatives since u and ρ depend only on x. The velocity derivative is approximated by

$$\frac{du}{dx} \simeq \frac{\Delta u}{\Delta x} = \frac{88 - 83}{0.204} = 25\frac{\text{m/s}}{\text{m}}$$

where the more accurate central difference has been used.[3] The density is

$$\rho = \frac{p}{RT} = \frac{345}{0.287 \times (10 + 273)} = 4.25\ \text{kg/m}^3$$

where absolute pressure and temperature are used. The density derivative is then approximated to be

$$\frac{d\rho}{dx} = -\frac{\rho}{u}\frac{du}{dx}$$
$$= -\frac{4.25\ \text{kg/m}^3}{86\ \text{m/s}} \times 25\ \text{s}^{-1} = \underline{-1.23\ \text{kg/m}^4}$$

5.2 Differential Continuity Equation

The x-component of velocity at points A, B, C, and D, which are 10 mm apart, is measured to be 5.76, 6.72, 7.61, and 8.47 m/s, respectively, in the plane steady, symmetrical, incompressible flow shown in Figure E5.3 in which $w = 0$. Approximate the x-component acceleration at C and the y-component of velocity 6 mm above B.

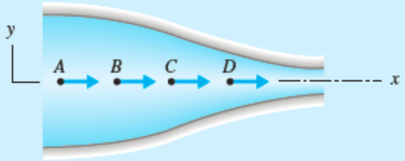


Figure E5.3

Solution

The desired acceleration component on the centerline is found from Eq. 3.2.9 to be

$$a_x = \cancelto{0}{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + \cancelto{0}{v\frac{\partial u}{\partial y}} + \cancelto{0}{w\frac{\partial u}{\partial z}}$$

$$\cong u\frac{\Delta u}{\Delta x} = 7.61\frac{8.47-6.72}{0.02} = \underline{666\ \text{m/s}^2}$$

where we have assumed a symmetrical flow so that v along the centerline is zero. We have used central differences to approximate $\partial u/\partial x$ at point C, as done in Example 5.2 (see footnote 3).

The y-component of velocity 6 mm above B is found using the continuity equation (5.2.10) as follows:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \qquad (\text{with } w = 0)$$

$$\frac{\Delta v}{\Delta y} \simeq -\frac{\Delta u}{\Delta x} = -\frac{7.61-5.76}{0.02} = -92.5$$

$$\therefore \Delta v = -92.5\ \Delta y = -92.5 \times 0.006 = -0.555\ \text{m/s}$$

We know that $v = 0$ at B; hence at the desired location, with $\Delta v = v - v_B$, there results

$$v = \underline{-0.555\ \text{m/s}}$$

5.2 Differential Continuity Equation

The continuity equation can be used to change the form of an expression. Write the expression $\rho D\tilde{u}/Dt + p\,\nabla \cdot \mathbf{V}$, which appears in the differential energy equation, in terms of enthalpy h rather than internal energy $\tilde{u}$. Recall that $h = \tilde{u} + p/\rho$ (see Eq. 1.7.11).

Solution

Using the definition of enthalpy, we can write

$$\frac{D\tilde{u}}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt}$$

where we used

$$\frac{D}{Dt}\left(\frac{p}{\rho}\right) = \frac{1}{\rho}\frac{Dp}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}$$

The desired expression is then

$$\rho\frac{D\tilde{u}}{Dt} + p\,\nabla \cdot \mathbf{V} = \rho\frac{Dh}{Dt} - \frac{Dp}{Dt} + \frac{p}{\rho}\frac{D\rho}{Dt} + p\,\nabla \cdot \mathbf{V}$$

The continuity equation (5.2.8) is introduced resulting in

$$\begin{aligned}\rho\frac{D\tilde{u}}{Dt} + p\,\nabla \cdot \mathbf{V} &= \rho\frac{Dh}{Dt} - \frac{Dp}{Dt} + \frac{p}{\rho}\frac{D\rho}{Dt} + p\left(-\frac{1}{\rho}\frac{D\rho}{Dt}\right)\\ &= \rho\frac{Dh}{Dt} - \frac{Dp}{Dt}\end{aligned}$$

and enthalpy has been introduced.

5.3 Differential Momentum Equation

5.3.1 General Formulation

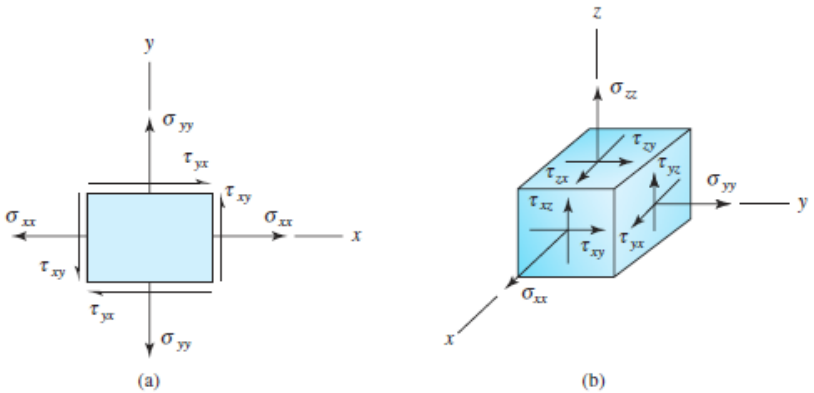
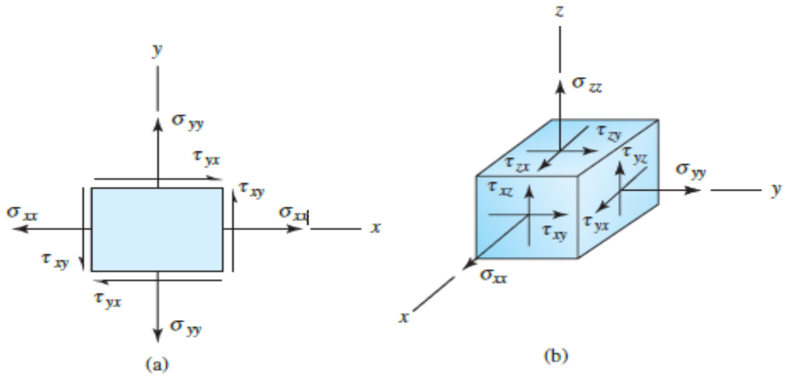


Figure 5.2 Stress components in rectangular coordinates: (a) two-dimensional stress components; (b) three-dimensional stress components.

- There are **nine** stress components of a **stress tensor τ_{ij}**.
- The stress components act in the positive direction on a positive face (normal vector points in the positive coordinate direction) and in the negative direction on a negative face (normal vector points in the negative coordinate direction).

5.3 Differential Momentum Equation

5.3.1 General Formulation



- **Normal Stress:** Stress component that acts perpendicular to a face (σ_{xx}, σ_{yy}, σ_{zz}).
- **Shear Stress:** Stress component that acts tangential to a face (τ_{xy}, τ_{yx}, τ_{xz}, τ_{zx}, τ_{yz}, τ_{zy}).

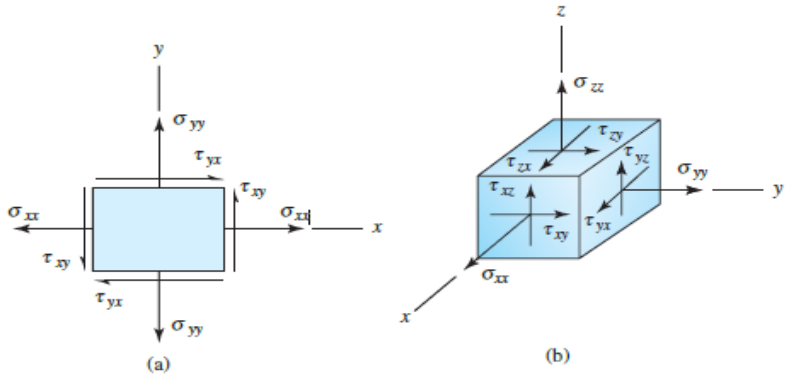
- Stress Tensor:

$$\tau_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

- First subscript on a stress component: Face upon which the component acts.
- Second subscript: Direction in which it acts.
  - E.g. τ_{xy} acts in the positive y-direction on a positive x-face and in the negative y-direction on a negative x-face.

5.3 Differential Momentum Equation

5.3.1 General Formulation



- **Normal Stress:** Stress component that acts perpendicular to a face (σ_{xx}, σ_{yy}, σ_{zz}).
- **Shear Stress:** Stress component that acts tangential to a face (τ_{xy}, τ_{yx}, τ_{xz}, τ_{zx}, τ_{yz}, τ_{zy}).

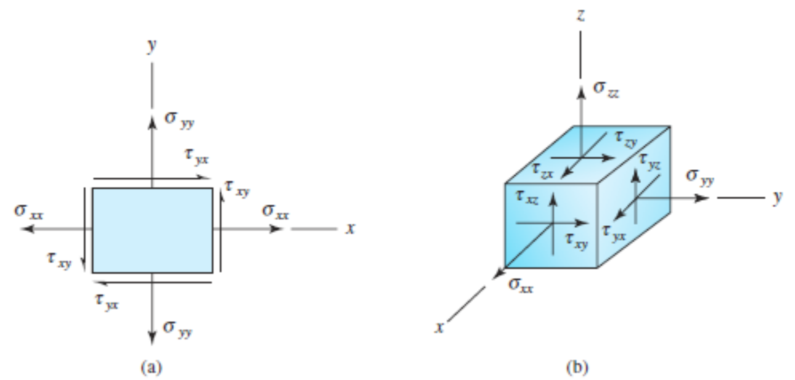
- Stress Tensor:

$$\tau_{ij} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

- This stress tensor is **symmetric**.
- $\tau_{xy} = \tau_{yx}$
- $\tau_{xz} = \tau_{zx}$
- $\tau_{yz} = \tau_{zy}$

5.3 Differential Momentum Equation

5.3.1 General Formulation



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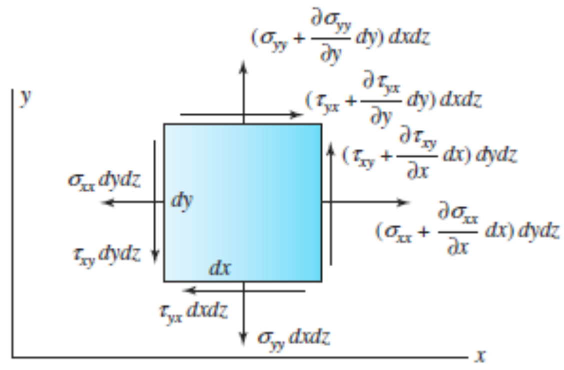


Figure 5.3 Rectangular stress components on a fluid element.

- Applying Newton’s second law:
  - Assuming no shear stress in the z-direction.

5.3 Differential Momentum Equation

5.3.1 General Formulation

- Applying Newton's second law:
  - Assuming no shear stress in the z-direction.

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx\right)dydz - \sigma_{xx}dydz + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}dy\right)dxdz - \tau_{xy}dxdz = \rho dxdydz\frac{Du}{Dt}$$

$$\left(\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y}dy\right)dxdz - \sigma_{yy}dxdz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial x}dx\right)dydz - \tau_{yx}dydz = \rho dxdydz\frac{Dv}{Dt}$$

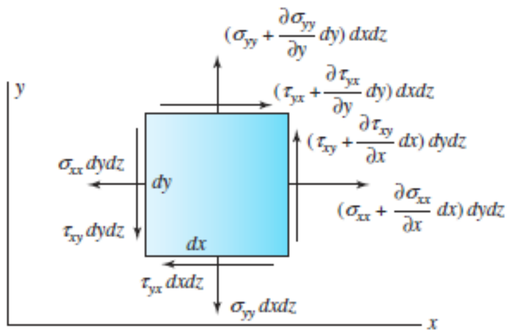


Figure 5.3 Rectangular stress components on a fluid element.

simplifies to:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \rho\frac{Du}{Dt}$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} = \rho\frac{Dv}{Dt}$$

z-direction components are included:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \rho\frac{Du}{Dt}$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = \rho\frac{Dv}{Dt}$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} - \rho g = \rho\frac{Dw}{Dt}$$

Assume gravity ρgdxdydz is in the negative z-direction

5.3 Differential Momentum Equation

5.3.2 Euler's Equations

$$\tau_{ij} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

- The stress tensor serves as a good approximation for flows away from boundaries, or in regions of sudden change.
  - Assume shear stress components (from viscous effects) are negligible.
  - Normal stress components are equal to the negative of the pressure.

5.3 Differential Momentum Equation

5.3.2 Euler's Equations

- For a frictionless flow, the stress components lead to:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho g_x$$
$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho g_y$$
$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho g_z$$

- The scalar equation can be written as a general vector equation as:

$$\rho \frac{D}{Dt}(u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}) = -\left(\frac{\partial p}{\partial x}\hat{\mathbf{i}} + \frac{\partial p}{\partial y}\hat{\mathbf{j}} + \frac{\partial p}{\partial z}\hat{\mathbf{k}}\right) - \rho \mathbf{g}$$

- Hence **Euler's Equation**

$$\rho \frac{D\mathbf{V}}{Dt} = -\boldsymbol{\nabla} p - \rho \mathbf{g}$$

Three differential equations formed from applying Newton's second law and neglecting viscous effects.

5.3 Differential Momentum Equation

A velocity field is proposed to be

$$u = \frac{10y}{x^2 + y^2} \qquad v = -\frac{10x}{x^2 + y^2} \qquad w = 0$$

(a) Is this a possible incompressible flow? (b) If so, find the pressure gradient ∇p assuming a frictionless air flow with the z-axis vertical. Use $\rho = 1.23$ kg/m^3.

Solution

(a) The continuity equation (5.2.10) is used to determine if the velocity field is possible. For this incompressible flow we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancelto{0}{\frac{\partial w}{\partial z}} = 0$$

Substituting in the velocity components, we have

$$\frac{\partial}{\partial x}\left(\frac{10y}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(-\frac{10x}{x^2 + y^2}\right) = \frac{-10y(2x)}{(x^2 + y^2)^2} - \frac{-10x(2y)}{(x^2 + y^2)^2} = \frac{1}{(x^2 + y^2)^2}[-20xy + 20xy] = 0$$

The quantity in brackets is obviously zero; hence the velocity field given is a possible incompressible flow.

5.3 Differential Momentum Equation

(b) The pressure gradient is found using Euler's equation. In component form we have the following:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho \cancelto{0}{g_x}$$

$$\therefore \frac{\partial p}{\partial x} = -\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\cancelto{0}{\frac{\partial u}{\partial z}} + \cancelto{0}{\frac{\partial u}{\partial t}}\right]$$

$$= -1.23\left[\frac{10y}{x^2+y^2}\frac{-20xy}{(x^2+y^2)^2} + \frac{-10x}{x^2+y^2}\frac{(x^2+y^2)10-10y(2y)}{(x^2+y^2)^2}\right]$$

$$= \frac{123x}{(x^2+y^2)^2}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho \cancelto{0}{g_y}$$

$$\therefore \frac{\partial p}{\partial y} = -\rho\left[u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\cancelto{0}{\frac{\partial v}{\partial z}} + \cancelto{0}{\frac{\partial v}{\partial t}}\right]$$

$$= -1.23\left[\frac{10y}{x^2+y^2}\frac{(x^2+y^2)(-10)+10x(2x)}{(x^2+y^2)^2} + \frac{-10x}{x^2+y^2}\frac{20xy}{(x^2+y^2)^2}\right]$$

$$= \frac{123y}{(x^2+y^2)^2}$$

$$\rho \cancelto{0}{\frac{Dw}{Dt}} = -\frac{\partial p}{\partial z} + \rho g_z$$

$$\therefore \frac{\partial p}{\partial z} = \rho g_z = 1.23 \text{ kg/m}^3 \times (-9.81) \text{ m/s}^2 = -12.07 \text{ N/m}^3$$

Thus $\nabla p = \frac{\partial p}{\partial x}\hat{\mathbf{i}} + \frac{\partial p}{\partial y}\hat{\mathbf{j}} + \frac{\partial p}{\partial z}\hat{\mathbf{k}} = \frac{123}{(x^2+y^2)^2}(x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) - 12.07\hat{\mathbf{k}} \text{ N/m}^3$

5.3 Differential Momentum Equation

Assume a steady, constant-density flow and integrate Euler's equation along a streamline in a plane flow.

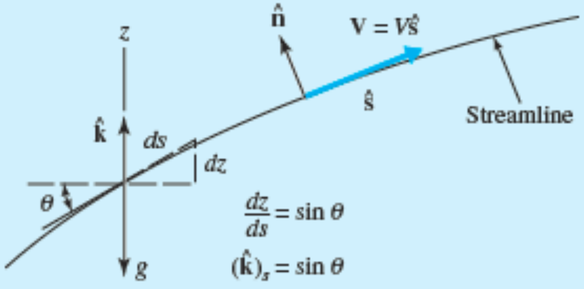


Figure E5.6

Solution

First, let us express the substantial derivative in streamline coordinates. Since the velocity vector is tangent to the streamline, we can write

$$\mathbf{V} = V\hat{\mathbf{s}}$$

where $\hat{\mathbf{s}}$ is the unit vector tangent to the streamline and V is the magnitude of the velocity, as shown in Figure E5.6. The substantial derivative is then, for this plane flow,

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + V\frac{\partial (V\hat{\mathbf{s}})}{\partial s} + \overbrace{\cancel{(V)_n}}^{0}\frac{\partial \mathbf{V}}{\partial n} = \frac{\partial \mathbf{V}}{\partial t} + V\frac{\partial V}{\partial s}\hat{\mathbf{s}} + V^2\frac{\partial \hat{\mathbf{s}}}{\partial s}$$

The quantity $\partial\hat{\mathbf{s}}/\partial s$ results from the change of the unit vector $\hat{\mathbf{s}}$; the unit vector cannot change magnitude (it must always have a magnitude of 1), it can only change direction. Hence the derivative $\partial\hat{\mathbf{s}}/\partial s$ is in a direction normal to the streamline and does not enter the streamwise component equation. For a steady flow $\partial\mathbf{V}/\partial t = 0$. Consequently, in the streamwise direction, Euler's equation (5.3.9) takes the form

5.3 Differential Momentum Equation

$$\rho V \frac{\partial V}{\partial s} = -\frac{\partial p}{\partial s} - \rho g \frac{\partial z}{\partial s}$$

recognizing that the component of $\hat{\mathbf{k}}$ along the streamline can be expressed as $(\hat{\mathbf{k}})_s = \partial z/\partial s$ (see the sketch above). Note that we use partial derivatives in this equation since velocity and pressure also vary with the normal coordinate.

The equation above can be written, assuming constant density so that $\partial \rho/\partial s = 0$, as

$$\frac{\partial}{\partial s}\left(\rho \frac{V^2}{2} + p + \rho g z\right) = 0$$

Integrating along the streamline results in

$$\rho \frac{V^2}{2} + p + \rho g z = \text{const.}$$

or

$$\frac{V^2}{2} + \frac{p}{\rho} + g z = \text{const.}$$

This is, of course, Bernoulli's equation. We have integrated along a streamline assuming constant density, steady flow, negligible viscous effects, and an inertial reference frame, so it is to be expected that Bernoulli's equation will result.

5.3 Differential Momentum Equation

5.3.3 Navier-Stokes Equations

- Most fluids are **Newtonian fluids**.
  - Have a linear relationship between stress components and velocity gradients.
  - E.g., Water, oil, and air
- If a fluid is Newtonian (linearity), and **isotropic**
  - Fluid properties are independent of direction at a given position.
  - Hence stress components and velocity gradients can be related using two fluid properties:
    - Viscosity μ
    - Second coefficient of viscosity λ

5.3 Differential Momentum Equation

5.3.3 Navier-Stokes Equations

- The stress-velocity gradient relations/constitutive equations:

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} + \lambda \nabla \cdot \mathbf{V} \qquad \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y} + \lambda \nabla \cdot \mathbf{V} \qquad \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z} + \lambda \nabla \cdot \mathbf{V} \qquad \tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

5.3 Differential Momentum Equation

5.3.3 Navier-Stokes Equations

Stokes's Hypothesis

- Relationship between viscosity and the second coefficient of viscosity (for most gases and monatomic gases).

$$\lambda = -\frac{2}{3}\mu$$

- From this, it can be said that the negative average of the three normal stresses is equal to the pressure, i.e.,

$$-\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = p$$

5.3 Differential Momentum Equation

5.3.3 Navier-Stokes Equations

- Hence for a **homogeneous** fluid in an incompressible flow:
    - **Using the continuity equations**

Homogeneous fluid: A fluid whose properties are independent of position.

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

- These are the **NAVIER-STOKES EQUATIONS**.

5.3 Differential Momentum Equation

5.3.3 Navier-Stokes Equations

- The Navier-Stokes equations can be placed in vector form as:

$$\frac{Du}{Dt}\hat{\mathbf{i}} + \frac{Dv}{Dt}\hat{\mathbf{j}} + \frac{Dw}{Dt}\hat{\mathbf{k}} = \frac{D\mathbf{V}}{Dt}$$

$$\frac{\partial p}{\partial x}\hat{\mathbf{i}} + \frac{\partial p}{\partial y}\hat{\mathbf{j}} + \frac{\partial p}{\partial z}\hat{\mathbf{k}} = \boldsymbol{\nabla} p$$

$$\nabla^2 u\hat{\mathbf{i}} + \nabla^2 v\hat{\mathbf{j}} + \nabla^2 w\hat{\mathbf{k}} = \nabla^2 \mathbf{V}$$

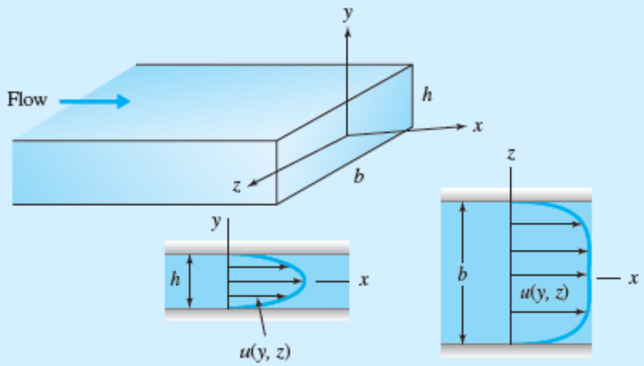
- With the Laplacian: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

- The combined vector equation for the Navier-Stokes is:

$$\boxed{\rho\frac{D\mathbf{V}}{Dt} = -\boldsymbol{\nabla} p + \rho \mathbf{g} + \mu\nabla^2 \mathbf{V}}$$

5.3 Differential Momentum Equation

Simplify the x-component Navier–Stokes equation for steady flow in a horizontal, rectangular channel assuming all streamlines parallel to the walls. Let the x-direction be in the direction of flow (Figure E5.7).



Solution

If the streamlines are parallel to the walls, only the x-component of velocity will be nonzero. Letting $v = w = 0$ the continuity equation (5.2.10) for an incompressible flow becomes

$$\frac{\partial u}{\partial x} = 0$$

showing that $u = u(y,z)$. The acceleration is then

$$\frac{Du}{Dt} = \cancelto{0}{\frac{\partial u}{\partial t}} + u\cancelto{0}{\frac{\partial u}{\partial x}} + \cancelto{0}{v}\frac{\partial u}{\partial y} + \cancelto{0}{w}\frac{\partial u}{\partial z} = 0$$

The x-component momentum equation then simplifies to

$$0 = -\frac{\partial p}{\partial x} + \cancelto{0}{\rho g_x} + \mu\left(\cancelto{0}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

since $\partial^2 u/\partial x^2 = 0$ if $\partial w/\partial x = 0$ and $g_x = 0$ for a horizontal channel. We then have

$$\frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

With the appropriate boundary conditions (the no-slip conditions), a solution to the foregoing equation could be sought. It would provide the velocity profiles sketched in Figure E5.7.

5.3 Differential Momentum Equation

5.3.4 Vorticity Equations

- Derived from taking the curl of the Navier-Stokes equations.
- Do not contain pressure or gravity terms in Navier-Stokes equations, only velocity.

$$\boldsymbol{\omega} = \nabla \times \mathbf{V}$$

$\nabla \times V$ is the curl of the velocity (cross-product of the del operator and a vector function.)

- Using this, the vorticity equation can be derived to be:

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{V} + \nu \nabla^2 \boldsymbol{\omega}$$

  - Assuming μ and ρ are constants.

5.3 Differential Momentum Equation

5.3.4 Vorticity Equations

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{V} + \nu \nabla^2 \boldsymbol{\omega}$$

- The vector form of the vorticity equation (above) can be rewritten as three scalar equations:

$$\frac{D\omega_x}{Dt} = \omega_x \frac{\partial u}{\partial x} + \omega_y \frac{\partial u}{\partial y} + \omega_z \frac{\partial u}{\partial z} + \nu \nabla^2 \omega_x$$

$$\frac{D\omega_y}{Dt} = \omega_x \frac{\partial v}{\partial x} + \omega_y \frac{\partial v}{\partial y} + \omega_z \frac{\partial v}{\partial z} + \nu \nabla^2 \omega_y$$

$$\frac{D\omega_z}{Dt} = \omega_x \frac{\partial w}{\partial x} + \omega_y \frac{\partial w}{\partial y} + \omega_z \frac{\partial w}{\partial z} + \nu \nabla^2 \omega_z$$

5.3 Differential Momentum Equation

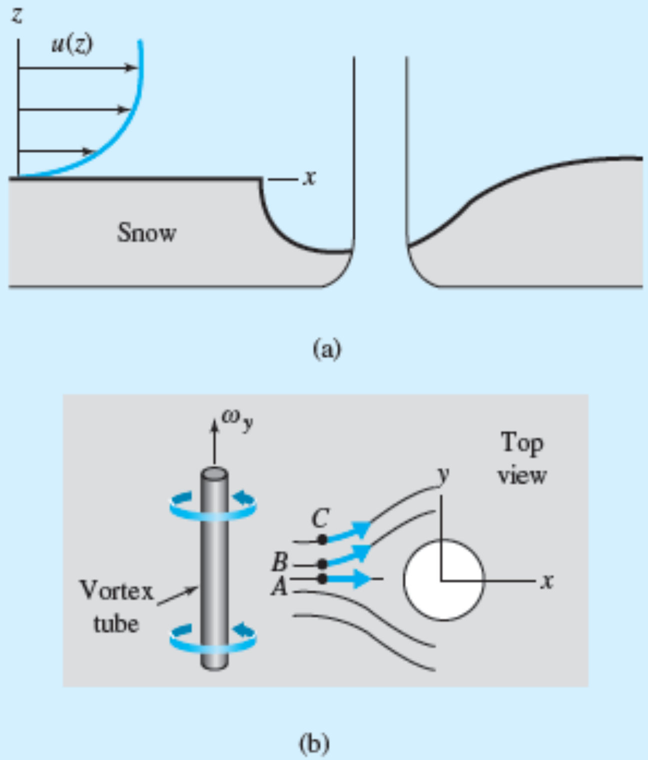
5.3.4 Vorticity Equations (Definitions)

- Vortex Line: A line to which the vorticity vector is tangent.
- Vortex Tube: (Vortex) A tube whose walls contain vortex lines.

- The vorticity equations show that if an inviscid flow is everywhere irrotational ($\omega = 0$):
    - It must remain irrotational since $\frac{D\omega}{Dt} = 0$ [Persistence of irrotationality]
- If a uniform flow approaches an object, vorticity is created because of viscosity.

5.3 Differential Momentum Equation

In a snowstorm, the snow is actually scooped out in front of a tree, or post, as shown in Figure E5.8a. Explain this phenomenon by referring to the vorticity equations.



5.3 Differential Momentum Equation

Solution

Let the velocity approaching the tree be in the x-direction with a velocity gradient $\partial u/\partial z$ near the ground. The vorticity components are then (refer to Eqs. 3.2.18)

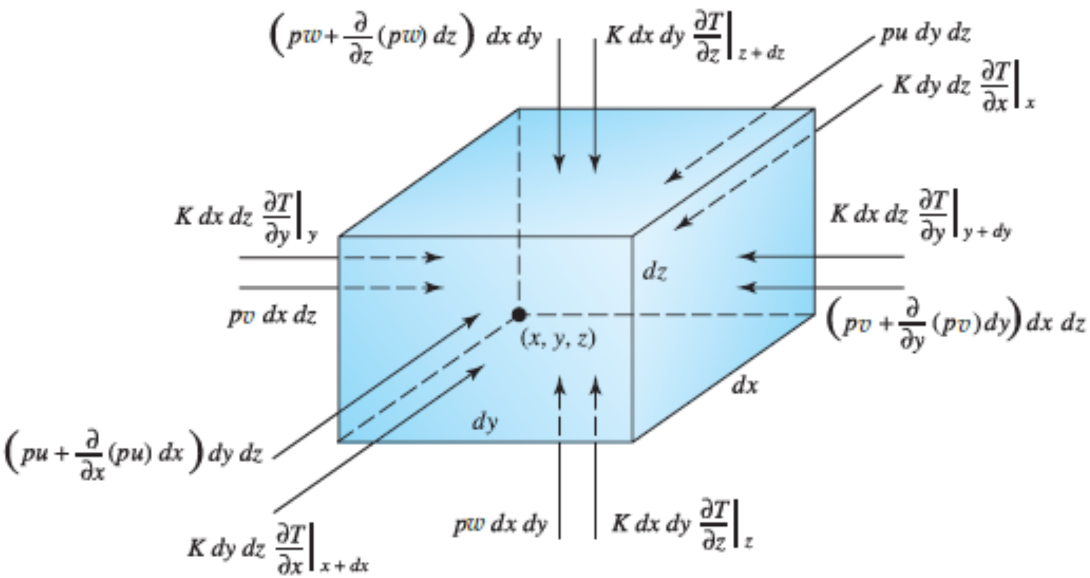
$$\omega_x = 0 \qquad \omega_y = \frac{\partial u}{\partial z} \qquad \omega_z = 0$$

The y-component vorticity equation (5.3.24), ignoring viscous effects over the short flow length, reduces to

$$\frac{D\omega_y}{Dt} = \omega_y \frac{\partial v}{\partial y}$$

Observe from Figure E5.8b that in the vicinity of the tree $\partial v/\partial y$ is positive since $v_C > v_B > v_A$. ($\partial v/\partial y$ can be shown to be positive for negative y also.) Since ω_y and $\partial v/\partial y$ are both positive, $D\omega_y/Dt$ is positive and ω_y increases as the vortex tubes approach the tree. This increased vorticity creates a strong vortex in front of the tree resulting in the snow being scooped out as shown. This same phenomenon occurs in a sandstorm or in a water flow around a post in a riverbed. Without the use of the verticity equation, it would be very difficult to explain.

5.4 Differential Energy Equation

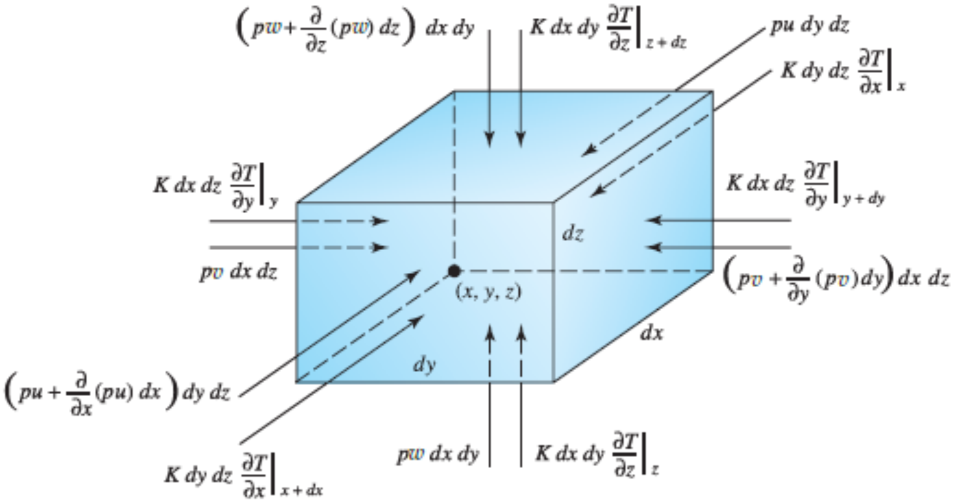


- For an infinitesimal fluid element, the heat transfer rate is:

$$\dot{Q} = -KA\frac{\partial T}{\partial n}$$

- Fourier's law of heat transfer,
  - n: Direction normal to the area
  - T: Temperature
  - K: Thermal conductivity

5.4 Differential Energy Equation



- The simplified energy equation is:

$$\rho \frac{D\tilde{u}}{Dt} = K\nabla^2 T - p\,\nabla \cdot \mathbf{V}$$

- In terms of enthalpy [$\tilde{u} = h - \frac{p}{\rho}$]

$$\rho \frac{Dh}{Dt} = K\nabla^2 T + \frac{Dp}{Dt}$$

5.4 Differential Energy Equation

- For a liquid flow with $\nabla \cdot V = 0$ and $\tilde{u} = c_p T$ (c_p is specific heat), the above simplifies to:

$$\frac{DT}{Dt} = \alpha \nabla^2 T$$

α is thermal diffusivity

$$\alpha = \frac{K}{\rho c_p}$$

- For an incompressible gas flow (and the ideal-gas assumption):

$$\rho c_p \frac{DT}{Dt} = K \nabla^2 T$$

5.4 Differential Energy Equation

A constant-density liquid flows into a wide, rectangular horizontal channel, the walls of which are maintained at a higher temperature than the liquid, as shown in Figure E5.9. Assume a variable μ, include viscous dissipation, and write the three describing differential equations for a steady flow.

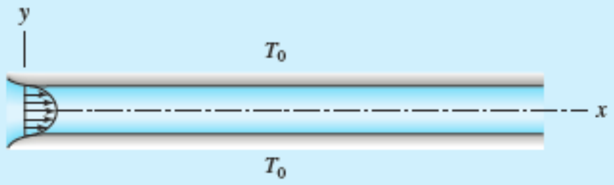


Figure E5.9

Solution

Let the x-axis coincide with centerline of the channel and the y-axis be vertical. The continuity equation would take the form

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

since $w = 0$ for the wide channel.

The flow will be primarily in the x-direction, but we must allow for variation of the y-component v. There will be no variation in the z-direction. The accelerations for this steady flow will be

$$\frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

$$\frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$

5.4 Differential Energy Equation

The stress terms contained in Eqs. 5.3.5 using Eqs. 5.3.10 with $\nabla \cdot \mathbf{V} = 0$, assuming a variable μ, become

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\frac{\partial \mu}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \mu}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + 2\frac{\partial \mu}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial \mu}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

The differential momentum equations (5.3.5) are then

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\frac{\partial \mu}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \mu}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + 2\frac{\partial \mu}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial \mu}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

The energy equation simplifies to

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{2\mu}{c_p}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]$$

where we have assumed K to be constant. The nonlinear, partial differential equations above, although quite formidable when attempting an analytic solution, could be solved numerically with the appropriate boundary conditions, and for a sufficiently low flow rate so that laminar flow exists (a turbulent flow is always unsteady and three-dimensional).

5.4 Differential Energy Equation

Show that for an ideal gas, $|Dp/Dt| \ll |p\,\nabla \cdot \mathbf{V}|$ in a low-speed flow, thereby concluding that Eq. 5.4.15 is the appropriate equation.

Solution

Let us consider a steady, uniform flow in a pipe so that $|\mathbf{V}| = u$ and $Dp/Dt = u\partial p/\partial x$. Then the problem can be stated as follows: Show that

$$\left|u\frac{\partial p}{\partial x}\right| \ll \left|p\frac{\partial u}{\partial x}\right|$$

Viscous effects are small and would not change the conclusion, so we can ignore any possible viscous effects. Then Euler's equation (5.3.7) allows us to use

$$\frac{\partial p}{\partial x} = -\rho u\frac{\partial u}{\partial x}$$

Using the definition of the speed of sound (Eq. 1.7.17) and the equation of state, we see that

$$c = \sqrt{\frac{kp}{\rho}} \quad \text{or} \quad p = c^2\frac{\rho}{k}$$

Thus

$$p\frac{\partial u}{\partial x} = \frac{c^2}{k}\rho\frac{\partial u}{\partial x}$$

Our problem can now be stated: Show that

$$\left|\rho u^2\frac{\partial u}{\partial x}\right| \ll \left|\frac{c^2}{k}\rho\frac{\partial u}{\partial x}\right|$$

Or, more simply, is it true that

$$u^2 \ll \frac{c^2}{k}\ ?$$

This can be seen to be true since we have assumed for a low-speed gas flow that the speed of the gas is much less than the speed of sound (e.g., $u < 0.3c$ or $\mathrm{M} < 0.3$). We know that k is of order unity ($k = 1.4$ for air), so it will not affect our conclusion that

$$\left|\frac{Dp}{Dt}\right| \ll |p\,\nabla \cdot \mathbf{V}|$$

5.5 Summary

- The vector form equations for incompressible flows are:

Continuity: $\nabla \cdot \mathbf{V} = 0$

Momentum: $\rho \dfrac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V}$

Energy: $\dfrac{DT}{Dt} = \alpha \nabla^2 T + \Phi / \rho c_p$ Liquids

Energy: $\rho c_p \dfrac{DT}{Dt} = K \nabla^2 T + \Phi$ Incompressible gases

Vorticity: $\dfrac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{V} + \nu \nabla^2 \boldsymbol{\omega}$

5.5 Summary (Definitions)

- **Newtonian Fluid** – One with a linear relationship between stress components and velocity gradients.
- **Isotropic Fluid** – One where the fluid properties are independent of direction.
- **Homogeneous Fluid** – One where fluid properties do not depend on position.
- **Incompressible Flow** – One where the density of a particle is constant ($\frac{D\rho}{Dt} = 0$)
- The vector form equations of the previous slide assumes all of the above, in addition to an *inertial reference frame*.

5.5 Summary (Fundamental Laws - Continuity)

Continuity

Rectangular

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Cylindrical

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Spherical

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} = 0$$

5.5 Summary (Fundamental Laws - Momentum)

Momentum

Rectangular

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g_x + \nu\nabla^2 u$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + g_y + \nu\nabla^2 v$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z + \nu\nabla^2 w$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Cylindrical

$$\frac{Dv_r}{Dt} - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r + \nu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)$$

$$\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + g_\theta + \nu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right)$$

$$\frac{Dv_z}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z + \nu\nabla^2 v_z$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Spherical

$$\frac{Dv_r}{Dt} - \frac{v_\theta^2 + v_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r + \nu\left(\nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2v_\theta \cot\theta}{r^2} - \frac{2}{r^2 \sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)$$

$$\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta - v_\phi^2 \cot\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + g_\theta + \nu\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2 \sin^2\theta} - \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right)$$

$$\frac{Dv_\phi}{Dt} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi \cot\theta}{r} = -\frac{1}{\rho r \sin\theta}\frac{\partial p}{\partial \phi} + g_\phi + \nu\left(\nabla^2 v_\phi - \frac{v_\phi}{r^2 \sin^2\theta} + \frac{2}{r^2 \sin^2\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial}{\partial \phi}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

5.5 Summary (Fundamental Laws - Energy)

Energy

Rectangular

$$\rho\frac{Dh}{Dt} = K\nabla^2 T + 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

Cylindrical

$$\rho\frac{Dh}{Dt} = K\nabla^2 T + 2\mu\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)^2\right]$$

Spherical

$$\rho\frac{Dh}{Dt} = K\nabla^2 T + 2\mu\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right)^2\right]$$
$$+ \mu\left[\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\theta}{\sin\theta}\right)\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right)^2 + \left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)^2\right]$$

5.5 Summary (Fundamental Laws - Stresses)

Stresses

Rectangular

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} \qquad \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y} \qquad \tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial z} \qquad \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Cylindrical

$$\sigma_{rr} = -p + 2\mu\frac{\partial v_r}{\partial r} \qquad \tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\sigma_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \qquad \tau_{\theta z} = \mu\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial v_z}{\partial z} \qquad \tau_{rz} = \mu\left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]$$

Spherical

$$\sigma_{rr} = -p + 2\mu\frac{\partial v_r}{\partial r}$$

$$\sigma_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)$$

$$\sigma_{\phi\phi} = -p + 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right)$$

$$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta\phi} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]$$

$$\tau_{r\phi} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$