Chapter 2

Potential Flow Theory

8.5 Potential-Flow Theory

8.5.1 Basic Flow Equations

- Inviscid flows exist outside the boundary layer and the wake in high-Reynolds number flows.
  - An airfoil has a thin boundary layer, hence this inviscid flow provides a good approximation to the flow.
  - Flow solution is essential to predict lift/drag and possible separation points.

$$\mathbf{V} = \nabla\phi$$

V = velocity field
Φ = velocity potential function

- This velocity field is called a potential flow (irrotational flow).
  - Property: Vorticity (ω) is zero.
  - Vorticity is the curl of the velocity vector.

$$\boldsymbol{\omega} = \nabla \times \mathbf{V} = 0$$

8.5 Potential-Flow Theory

8.5.1 Basic Flow Equations

- The vorticity equation is obtained by taking the curl of the Navier-Stokes equation.

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\mathbf{V} + \nu \nabla^2 \boldsymbol{\omega}$$

- If ω is zero → $\frac{D\omega}{Dt}$ can only be nonzero if viscous effects act through the second term.
  - If viscous effects are absent (inviscid flow), the vorticity must be zero.

- With the velocity given by the scalar function gradient, for an incompressible flow:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \nabla^2\phi = 0 \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

- This is **Laplace's equation**.

8.5 Potential-Flow Theory

8.5.1 Basic Flow Equations

- Simplify by focusing on 2D flows.
  - Continuity equation: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
  - For velocity components u, v which depend on x and y:

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

- The continuity solution is satisfied.
  - Hence the function ψ(x,y) is called a **stream function**.
    - The stream function is constant along a streamline (dψ = 0).

8.5 Potential-Flow Theory

8.5.1 Basic Flow Equations

- The vorticity vector for a plane flow only has a z-component (w = 0)
  - No variation with z.

$$\omega_z = (\nabla \times \mathbf{V})_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0}$$

- The stream function (ψ) and potential function (Φ) satisfy Laplace's equation for a plane flow.

- Hence, the Cauchy-Riemann equations are given as:

$$u = \frac{\partial \Phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } v = \frac{\partial \Phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

8.5 Potential-Flow Theory

A scalar potential function is given by $\phi = A \tan^{-1}(y/x)$. Find the stream function $\psi(x, y)$.

Solution

The relationship between ϕ and ψ is given by Eq. 8.5.11. We have

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(A \tan^{-1}\frac{y}{x}\right) = -\frac{Ay}{x^2 + y^2}$$

This can be integrated as follows:

$$\int \frac{\partial \psi}{\partial y} dy = -A \int \frac{y}{x^2 + y^2} dy$$

$$\therefore \psi = -\frac{A}{2} \ln(x^2 + y^2) + f(x)$$

A function of x rather than a constant must be added since partial derivatives are being used. Now, let us differentiate this expression with respect to x. There results

$$\frac{\partial \psi}{\partial x} = -A \frac{x}{x^2 + y^2} + \frac{df}{dx}$$

This must equal $(-\partial \phi / \partial y)$ as demanded by Eq. 8.5.11; that is,

$$-\frac{Ax}{x^2 + y^2} + \frac{df}{dx} = -\frac{Ax}{x^2 + y^2}$$

Thus

$$\frac{df}{dx} = 0 \qquad \text{or} \qquad f = C$$

Since ϕ and ψ are used to find the velocity components by differentiation, the constant C is of no concern; it is usually set equal to zero. Hence

$$\psi = -\frac{A}{2} \ln(x^2 + y^2)$$

8.5 Potential-Flow Theory

Show that the difference in the stream function between any two streamlines is equal to the flow rate per unit depth between the two streamlines. The flow rate per unit depth is denoted by q.

Solution

Consider the flow between two streamlines infinitesimally close, as shown in Figure E8.9a. The flow rate per unit depth through the elemental area is, referring to Figure E8.9b,

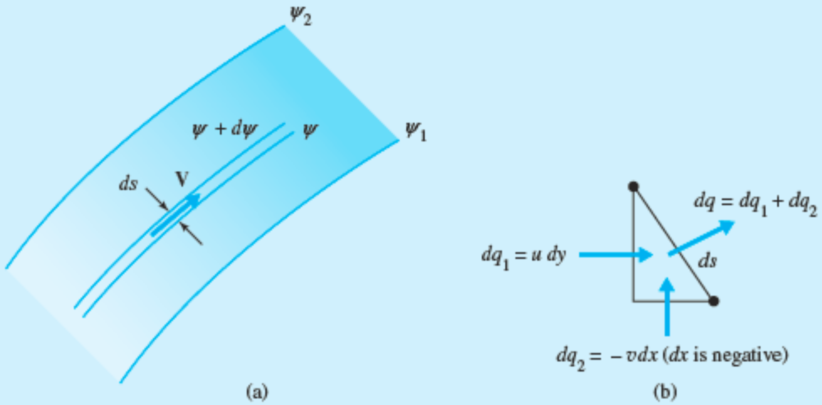


Figure E8.9

$$
\begin{aligned}
dq &= dq_1 + dq_2 \\
&= u dy - v dx \\
&= \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = d\psi
\end{aligned}
$$

If this is integrated between two streamlines with $\psi = \psi_1$ and $\psi = \psi_2$, there results

$$q = \psi_2 - \psi_1$$

thereby proving the statement of the example.

8.5 Potential-Flow Theory

Show that the streamlines and equipotential lines of a plane, incompressible, potential flow intersect one another at right angles.

Solution

If, at a point, the slope of a streamline is the negative reciprocal of the slope of an equipotential line, the two lines are perpendicular to each other. The slope of a streamline (see Figure E8.10a) is given by

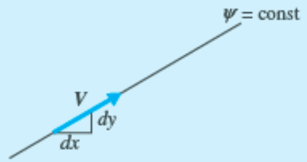


Figure E8.10a

$$\left.\frac{dy}{dx}\right|_{\psi = \text{const}} = \frac{v}{u}$$

The slope of an equipotential line is found from

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0$$

since ϕ = constant along an equipotential line. This gives

$$\left.\frac{dy}{dx}\right|_{\phi = \text{const}} = -\frac{\partial \phi/\partial x}{\partial \phi/\partial y} = -\frac{u}{v}$$

Hence we see that the slope of the streamline is the negative reciprocal of the slope of the equipotential line; that is,

$$\left.\frac{dy}{dx}\right|_{\phi = \text{const}} = -\left(\left.\frac{dy}{dx}\right|_{\psi = \text{const}}\right)^{-1}$$

8.5 Potential-Flow Theory

Thus, whenever the streamlines intersect the equipotential lines, they must do so at right angles. A sketch of the streamlines and equipotential lines (equally spaced at large distances from the body), known as a *flow net*, is shown in Figure E8.10b for flow over a weir. Such a carefully constructed sketch can be used to approximate the velocities at points of interest in an inviscid flow. Pressures can then be estimated using Bernoulli's equation.

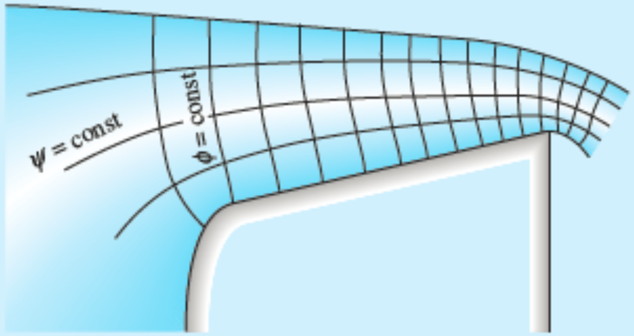


Figure E8.10b

8.5 Potential-Flow Theory

8.5.2 Simple Solutions

- Laplace's equation, the continuity equation, and the velocity components in polar form are:

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$$

$$\nabla\cdot\mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} = 0$$

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} \qquad v_\theta = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

Easier to manipulate in polar coordinates

8.5 Potential-Flow Theory

8.5.2 Simple Solutions

- There are four simple solutions to the Laplace flow.

Uniform flow:	$\psi = U_\infty y$	$\phi = U_\infty x$
Line source:	$\psi = \dfrac{q}{2\pi}\theta$	$\phi = \dfrac{q}{2\pi}\ln r$
Irrotational vortex:	$\psi = \dfrac{\Gamma}{2\pi}\ln r$	$\phi = \dfrac{\Gamma}{2\pi}\theta$
Doublet:	$\psi = -\dfrac{\mu \sin\theta}{r}$	$\phi = -\dfrac{\mu \cos\theta}{r}$

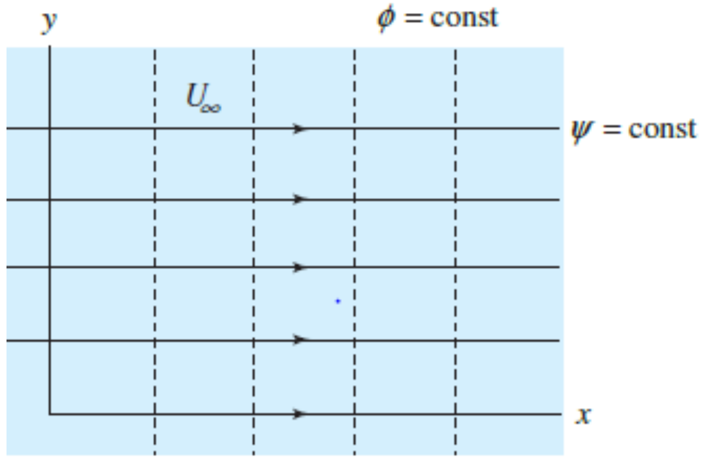
- The **uniform flow velocity (U_∞)** is assumed to be in the x-direction.
- **Source strength (q)** is the volume rate of flow per unit depth.
  - Positive value: issues from the source.
  - Negative value: creates a sink
- The **vortex strength (Γ)** is the circulation about the origin (clockwise = positive).

$$\Gamma = \oint_L \mathbf{V} \cdot d\mathbf{s}$$

L: Closed curve around the origin

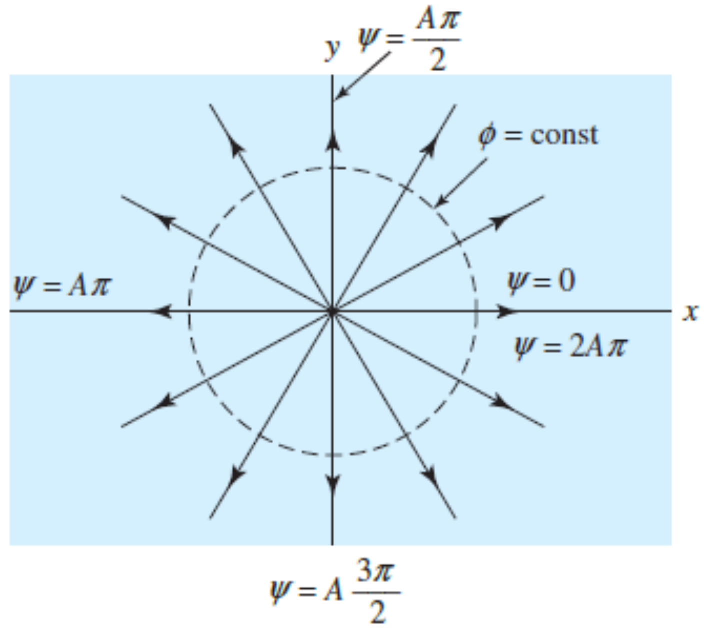
8.5 Potential-Flow Theory

8.5.2 Simple Solutions



(a) Uniform flow in x-direction

$$\text{Uniform flow:}\quad u = U_\infty \qquad v = 0$$
$$v_r = U_\infty \cos\theta \qquad v_\theta = -U_\infty \sin\theta$$

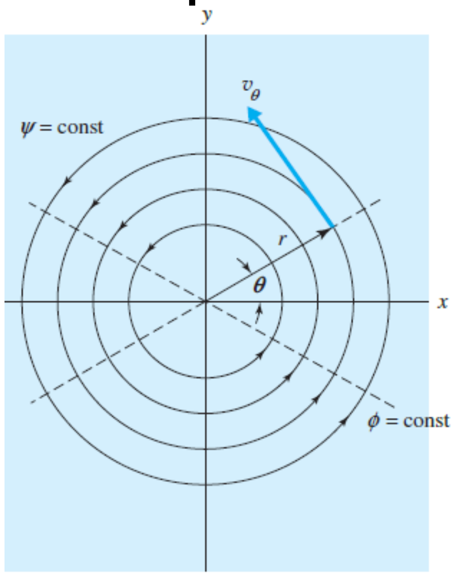


(b) Line source

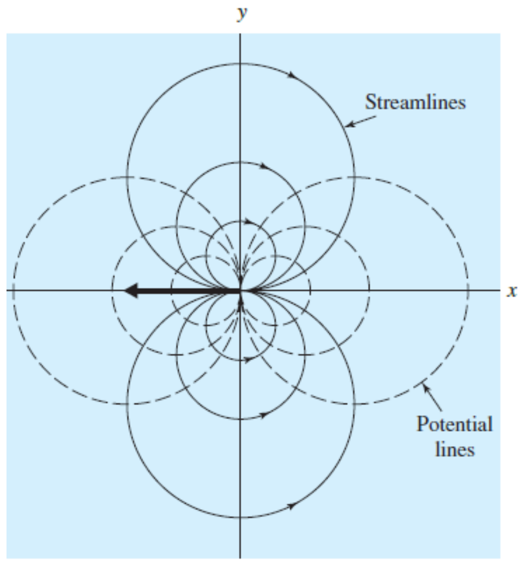
$$\text{Line source:}\quad v_r = \frac{q}{2\pi r} \qquad v_\theta = 0$$
$$u = \frac{q}{2\pi}\frac{x}{x^2 + y^2} \qquad v = \frac{q}{2\pi}\frac{y}{x^2 + y^2}$$

8.5 Potential-Flow Theory

8.5.2 Simple Solutions



(c) Irrotational vortex



(d) Doublet

Irrotational vortex : $v_r = 0 \qquad v_\theta = -\dfrac{\Gamma}{2\pi r}$

$$u = -\frac{\Gamma}{2\pi}\frac{y}{x^2 + y^2} \qquad v = \frac{\Gamma}{2\pi}\frac{x}{x^2 + y^2}$$

Found when water swirls down a drain/turbine of a hydropower dam

Doublet: $v_r = -\dfrac{\mu \cos\theta}{r^2} \qquad v_\theta = -\dfrac{\mu \sin\theta}{r^2}$

$$u = -\mu\frac{x^2 - y^2}{(x^2 + y^2)^2} \qquad v = -\mu\frac{2xy}{(x^2 + y^2)^2}$$

Doublet strength, μ is for a doublet oriented in the negative-x direction.

8.5 Potential-Flow Theory

The pressure far from an irrotational vortex (a simplified tornado) in the atmosphere is zero gage. If the velocity at $r = 20$ m is 20 m/s, estimate the velocity and the pressure at $r = 2$ m. (The irrotational vortex ceases to be a good model for a tornado when r is small. In the "eye" of the tornado the motion is approximated by rigid-body motion.)

Solution

For an irrotational vortex, we know that

$$v_\theta = -\frac{\Gamma}{2\pi r}$$

Hence

$$\begin{aligned}\Gamma &= -2\pi r v_\theta \\ &= -2\pi \times 20 \times 20 = -800\pi \text{ m}^2/\text{s}\end{aligned}$$

The velocity at $r = 2$ m is then

$$v_\theta = -\frac{-800\pi}{2\pi \times 2} = 200 \text{ m/s}$$

Bernoulli's equation for this incompressible, inviscid, steady flow gives the pressure as follows assuming a stagnant atmosphere away from the tornado:

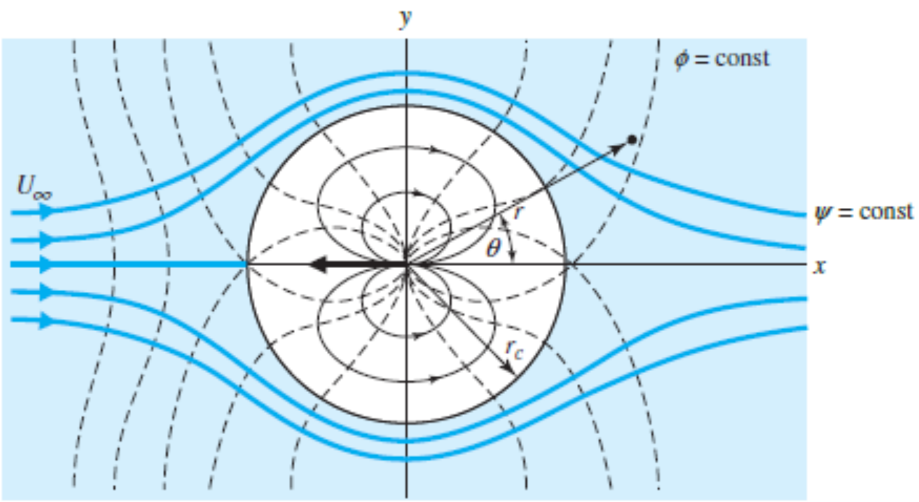
$$\begin{aligned}\cancelto{0}{p_\infty} + \cancelto{0}{\frac{U_\infty^2}{2}}\rho &= p + \frac{v_\theta^2}{2}\rho \\ \therefore p &= -\frac{1}{2}\rho v_\theta^2 \\ &= -\frac{1}{2} \times 1.20 \times 200^2 = -24000 \text{ Pa} \quad \text{or} \quad \underline{-24 \text{ kPa}}\end{aligned}$$

The negative sign denotes a vacuum. It is this vacuum that causes roofs of buildings to blow off during a tornado.

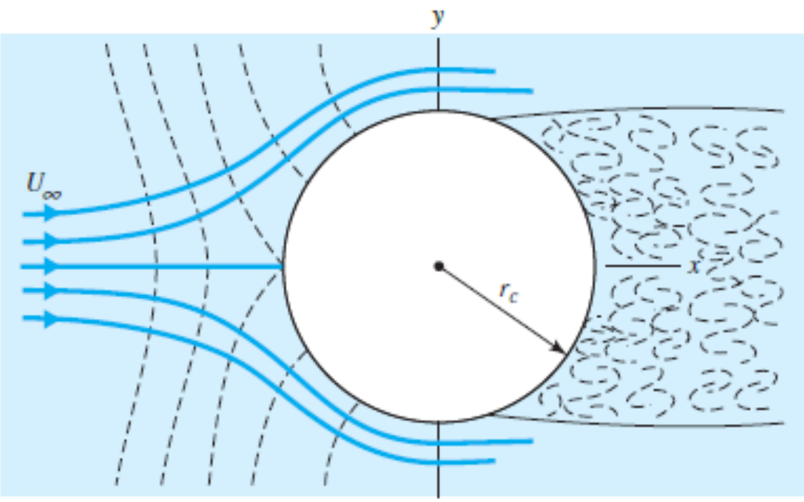
8.5 Potential-Flow Theory

8.5.3 Superposition

- The simple flows previously presented can be superimposed (with each other) to obtain more complicated flows.



(a) Potential flow



(b) Actual flow

8.5 Potential-Flow Theory

8.5.3 Superposition

- E.g., Flow around a circular cylinder with and without circulation.
  - **Superimpose a uniform flow and a doublet.**

$$\psi = U_\infty y - \frac{\mu \sin\theta}{r}$$

- The velocity component, v_r is:

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

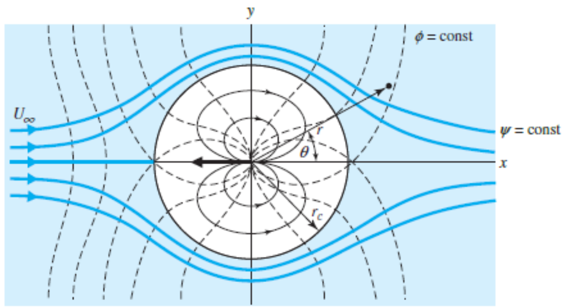
$$= U_\infty \cos\theta - \frac{\mu}{r^2}\cos\theta$$

- At the radius, r_c → v_r is zero.
  - Zero for all θ → hence the circle $r = r_c$ is a streamline.

$$r_c = \sqrt{\frac{\mu}{U_\infty}}$$

8.5 Potential-Flow Theory

8.5.3 Superposition



(a) Potential flow

- The stagnation points are where $v_\theta = 0$ on the circle $r = r_c$

$$v_\theta = -\frac{\partial \psi}{\partial r}$$
$$= -U_\infty \sin\theta - \frac{\mu \sin\theta}{r_c^2} = -2U_\infty \sin\theta = 0$$

- The stagnation points are at θ = 0° and θ = 180°.

- To find the pressure distribution, use Bernoulli's equation between stagnation point ($V = 0$; $p = p_0$) and an arbitrary point.

$$p_c = p_0 - \rho\frac{v_\theta^2}{2}$$
$$= p_0 - 2\rho U_\infty^2 \sin^2\theta$$

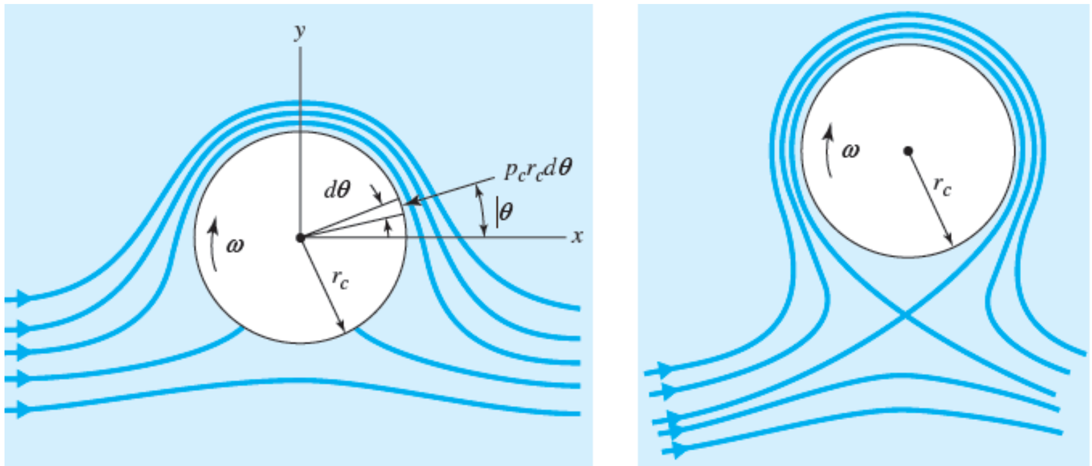
8.5 Potential-Flow Theory

8.5.3 Superposition

- E.g., Flow around a rotating cylinder.
    - **Add an irrotational vortex to the superimposed uniform flow and doublet.**

$$\psi = U_\infty y - \frac{\mu \sin\theta}{r} + \frac{\Gamma}{2\pi} \ln r$$

- The vortex flow (consisting of circular streamlines) does not change v_r → the cylinder $r = r_c$ is unchanged.
- Stagnation points change.



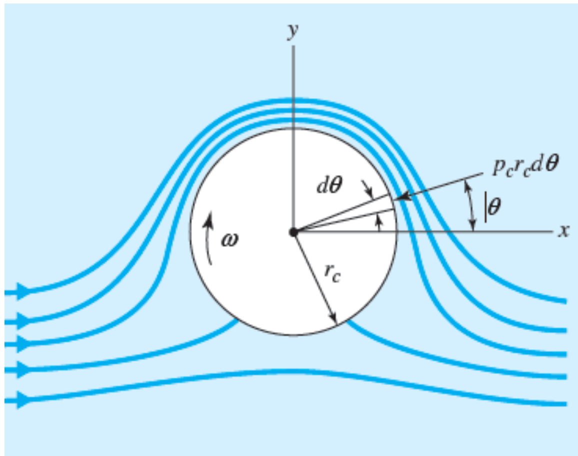
(a) $\Gamma < 4\pi U_\infty r_c$ (b) $\Gamma > 4\pi U_\infty r_c$

8.5 Potential-Flow Theory

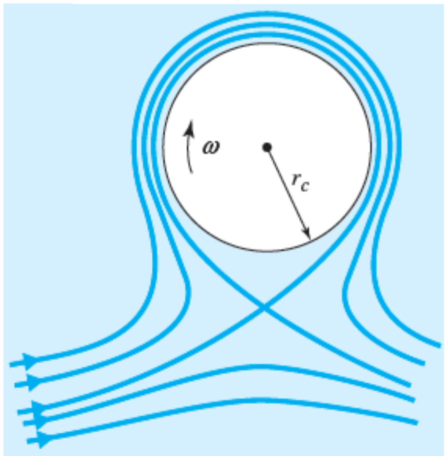
8.5.3 Superposition

- The stagnation points are:

$$v_\theta = -\frac{\partial \psi}{\partial r}$$
$$= -2U_\infty \sin\theta - \frac{\Gamma}{2\pi r_c} = 0$$



(a) $\Gamma < 4\pi U_\infty r_c$



(b) $\Gamma > 4\pi U_\infty r_c$

For (a), the stagnation points are on the cylinder where $r = r_c$.

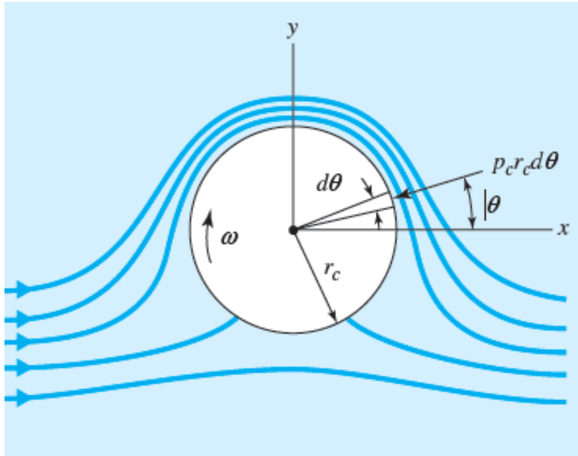
For (b), the circulation is large so that a single stagnation point ($\theta = 270°$) is formed off the cylinder.

8.5 Potential-Flow Theory

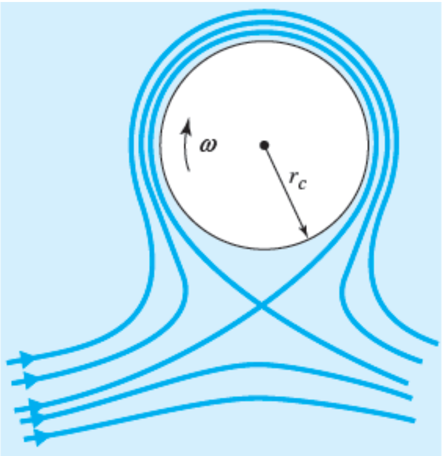
8.5.3 Superposition

- From Bernoulli's equation, the pressure distribution is:

$$p_c = p_0 - \rho \frac{U_\infty^2}{2}\left(2\sin\theta + \frac{\Gamma}{2\pi r_c U_\infty}\right)^2$$



(a) $\Gamma < 4\pi U_\infty r_c$



(b) $\Gamma > 4\pi U_\infty r_c$

- After integration, drag = 0 and the lift per unit length is:

$$F_L = -\int_0^{2\pi} p_c \sin\theta\, r_c\, d\theta = \rho U_\infty \Gamma$$

- Good approximation to the lift for all cylinders (and airfoils).

- **KUTTA-JOUKOWSKY THEOREM!**

8.5 Potential-Flow Theory

An 200-mm-diameter cylinder rotates clockwise at 1000 rpm in a 15°C-atmospheric airstream flowing at 4.5 m/s. Locate any stagnation points and find the minimum pressure on the cylinder.

Solution

The circulation is calculated (see Eq. 8.5.19) to be

$$\Gamma = \oint_L \mathbf{V}\cdot d\mathbf{s}$$

$$= 2\pi r_c^2\omega = 2\pi \times (0.1)^2 \times \frac{1000 \times 2\pi}{60} = 6.57 \text{ m}^2/\text{s}$$

This is greater than $4\pi U_\infty r_c = 4\pi \times 4.5 \times 0.1 = 5.65 \text{ m}^2/\text{s}$; hence the stagnation point is off the cylinder (see Figure 8.20b) at $\underline{\theta = 270°}$ at a radius of

$$r_0 = -\frac{\Gamma}{4\pi U_\infty \sin 270°} = -\frac{6.57}{4\pi \times 4.5 \times (-1)} = 0.116 \text{ m}$$

Only one stagnation point exists.

The minimum pressure is located on the top of the cylinder where $\theta = 90°$. Using Bernoulli's equation from the free stream to that point, we have, letting $p_\infty = 0$,

$$\cancelto{0}{p_\infty} + \rho\frac{U_\infty^2}{2} = p_{\min} + \frac{\rho}{2}(v_\theta)_{\max}^2$$

$$\therefore p_{\min} = \frac{\rho}{2}[U_\infty^2 - (v_\theta)_{\max}^2] = \frac{\rho}{2}\left[U_\infty^2 - \left(-2U_\infty \sin 90° - \frac{\Gamma}{2\pi r_c}\right)^2\right]$$

$$= \frac{1.2 \text{ kg/m}^3}{2}\left[4.5^2 - \left(2 \times 4.5 + \frac{6.57}{2\pi \times 0.1}\right)^2\right] \text{m}^2/\text{s}^2 = \underline{-215.1 \text{ Pa}}$$

8.7 Summary

- Drag and Lift coefficients are: $C_D = \dfrac{\text{Drag}}{\frac{1}{2}\rho V^2 A}$ $\quad C_L = \dfrac{\text{Lift}}{\frac{1}{2}\rho V^2 A}$
- Vortex shedding occurs from a cylinder when 300 < Re < 10,000
  - The frequency of shedding is found from the Strouhal number. $\mathrm{St} = \dfrac{fD}{V}$
- Plane potential flows can be found by superimposing simple flows below.

Uniform flow:	$\psi = U_\infty y$	$\phi = U_\infty x$
Line source:	$\psi = \dfrac{q}{2\pi\theta}$	$\phi = \dfrac{q}{2\pi}\ln r$
Irrotational vortex:	$\psi = \dfrac{\Gamma}{2\pi}\ln r$	$\phi = \dfrac{\Gamma}{2\pi}\theta$
Doublet:	$\psi = -\dfrac{\mu}{r}\sin\theta$	$\phi = -\dfrac{\mu}{r}\cos\theta$

- The stream function for a rotating cylinder is: $\psi_{\text{cylinder}} = U_\infty y - \dfrac{\mu}{r}\sin\theta + \dfrac{\Gamma}{2\pi}\ln r$
  - With the cylinder radius: $r_c = \sqrt{\dfrac{\mu}{U_\infty}}$

8.7 Summary

- The velocity components are:

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

- For a laminar boundary layer on a flat plate (zero pressure gradient), the exact solution is:

$$\delta = 5\sqrt{\frac{\nu x}{U_\infty}} \qquad c_f = 0.664\sqrt{\frac{\nu}{xU_\infty}} \qquad C_f = 1.33\sqrt{\frac{\nu}{LU_\infty}}$$

- For a turbulent flow, the power-law profile (η = 7):

$$\delta = 0.38x\left(\frac{\nu}{xU_\infty}\right)^{1/5} \qquad c_f = 0.059\left(\frac{xU_\infty}{\nu}\right)^{1/5} \qquad C_f = 0.073\left(\frac{\nu}{LU_\infty}\right)^{1/5}$$

  - The wall shear and drag force per unit width are:

$$\tau_0 = \frac{1}{2}c_f \rho U_\infty^2 \qquad F_D = \frac{1}{2}C_f \rho U_\infty^2 L$$