Chapter 3

Boundary layer

8.1 Introduction

- An understanding of external flows is important for aerospace engineers:
  - To understand airflows around different components of an aircraft.
- Also for flow of fluids around turbines, buildings, automobiles, etc.
- Need to concentrate on high-Reynolds-number flows (Re > 1000).
  1. Incompressible immersed flows (automobiles, low-speed aircraft, turbines)
  2. Flows of liquids with a free surface (ship or bridge abutments)
  3. Compressible flows with high-speed objects
- Flow is influenced by the presence of a boundary or another object.

8.1 Introduction

- High-Reynolds-number incompressible immersed flows can be either:
  - Flows around blunt bodies.
  - Flows around streamlined bodies.

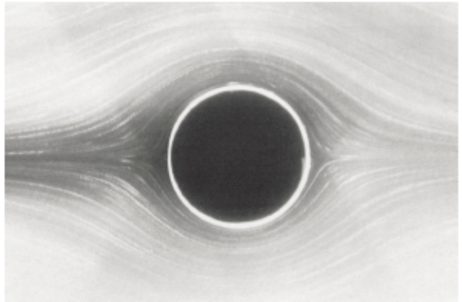


Figure 8.1 Flow past a circular cylinder at Re = 0.16. The flow is from left to right. It resembles superficially the pattern of potential flow. The flow of water is shown by aluminum dust. (Photograph by Sadatoshi Taneda. From *Album of Fluid Motion*, 1982, The Parabolic Press, Stanford, California.)

lbl = Laminar boundary layer
tbl = Turbulent boundary layer

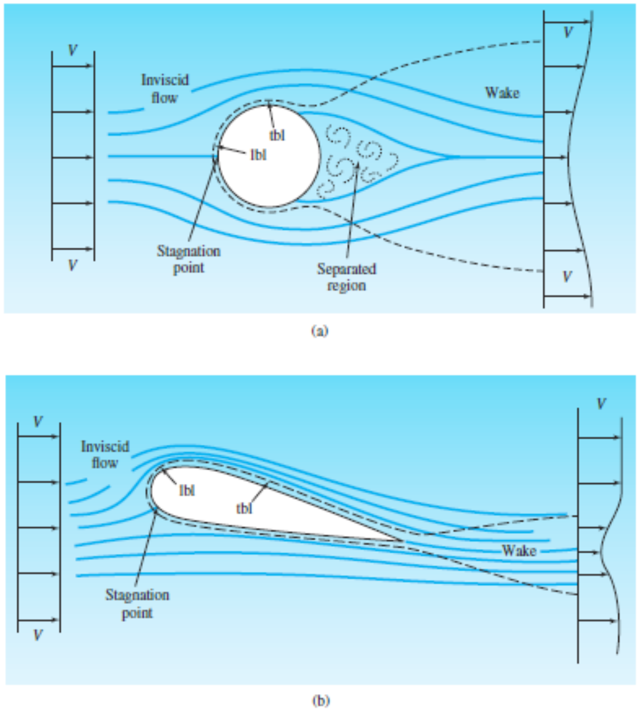
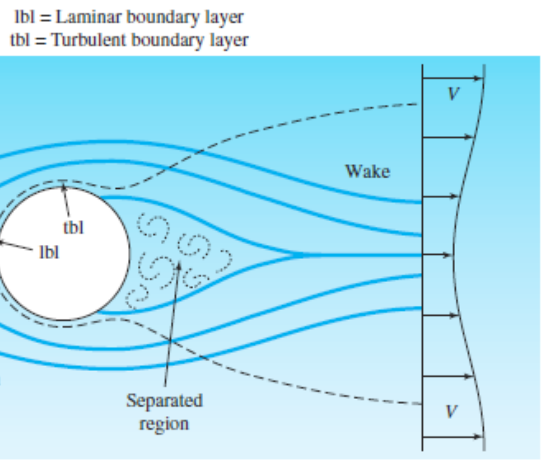


Figure 8.3 Flow around a blunt body and a streamlined body.

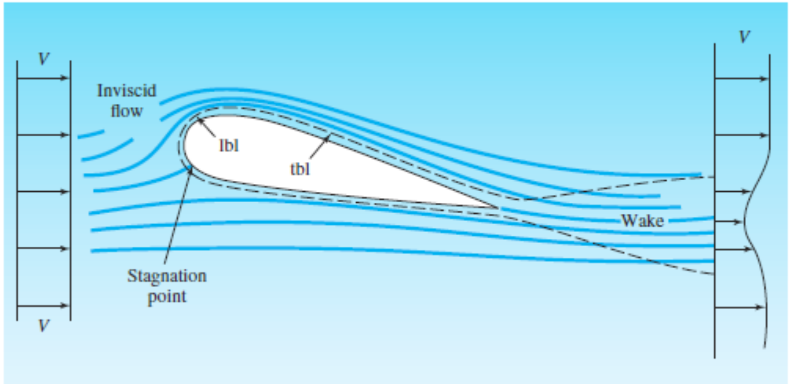
8.1 Introduction

- Boundary layer near a stagnation point is a laminar boundary layer.
- For a high enough Reynolds number, there is a laminar-turbulent transition downstream:
  - Flow may separate from the body, forming a separated region [region of recirculating flow].
  - The wake is a region of velocity defect that grows because of diffusion.
  - **These boundaries are time-dependent.**



(a)

8.1 Introduction



(b)

- For the figure above, the drag and the lift is found as below:
  - Drag: Force the flow exerts on a body in the direction of the flow.
  - Lift: Force the flow exerts normal to the direction of flow.

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} \qquad C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A}$$

- The lift and drag coefficients are seen above, with *A* as the projected area.

8.2 Separation

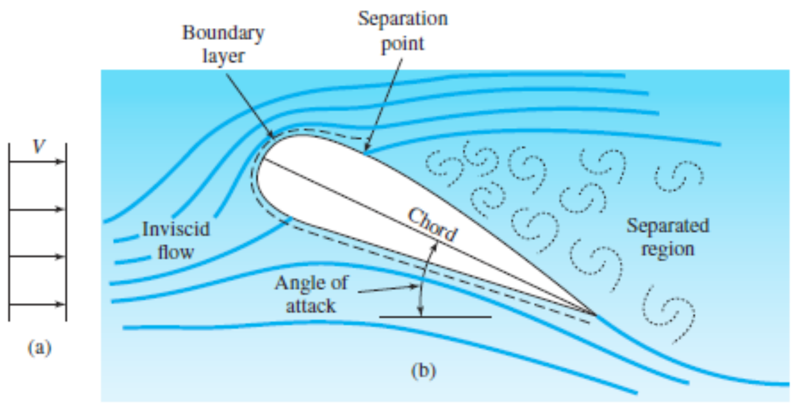


Figure 8.4 Streamlined body that is stalled.

Chord: Line connecting the trailing edge with the nose
Angle of attack: Angle the flow makes with the chord

- A separated flow occurs when the main stream flow leaves the body.
- When separation occurs with a high angle of attack near the forward portion of the airfoil, the flow is "stalled."
- Stall is undesirable in aircrafts while cruising, but provides high drag when landing.

8.2 Separation

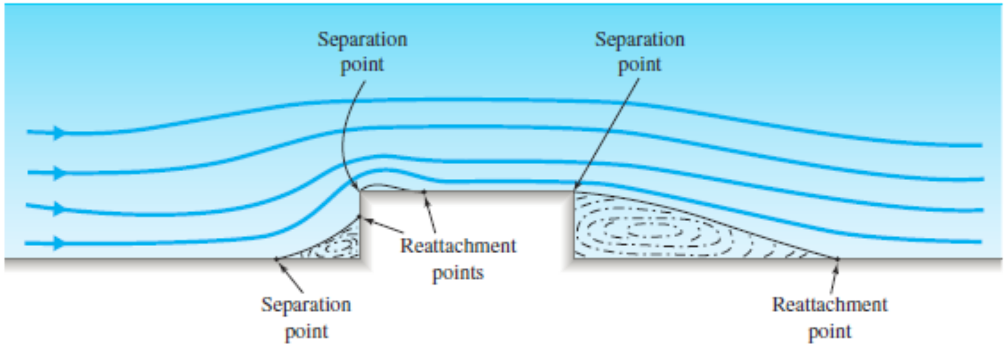


Figure 8.5 Separation due to abrupt geometry changes.

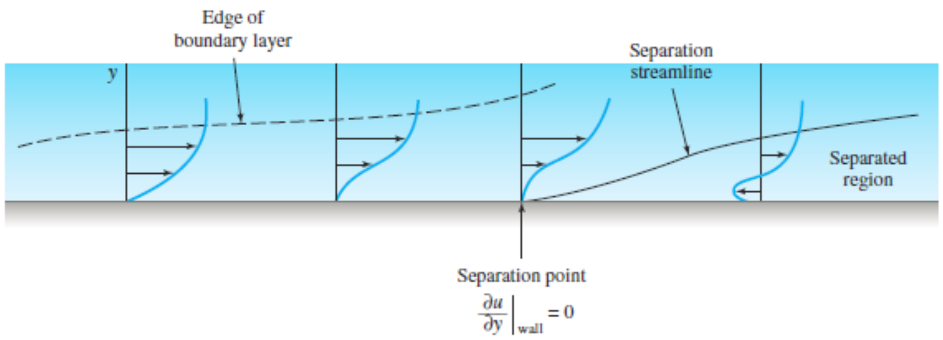


Figure 8.6 Flow separation on a flat surface due to an adverse pressure gradient.

- If a body has an abrupt change in profile, separation occurs near this change.
- It will also occur upstream of the flat surface, and will reattach at some point downstream.

8.2 Separation

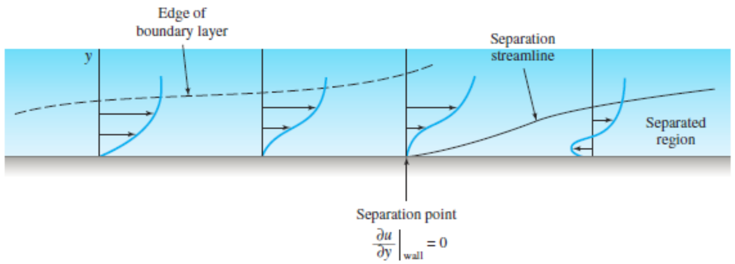


Figure 8.6 Flow separation on a flat surface due to an adverse pressure gradient.

- The y-coordinate is normal to the wall. The x-coordinate is measured along the wall.
- Downstream of the separation point, the x-component velocity near the wall is in the negative x-direction. Upstream of the separation point, the x-component of the velocity is in the positive x-direction.
  - Separation point is at the point where $(\partial u/\partial y)_{\text{wall}} = 0$

- Separation occurs when the flow approaches a stagnation region:
  - Velocity is low and pressure is high (positive pressure gradient).
  - As separation is undesirable, a positive pressure gradient is an adverse pressure gradient. A negative gradient is a favorable pressure gradient.

8.2 Separation

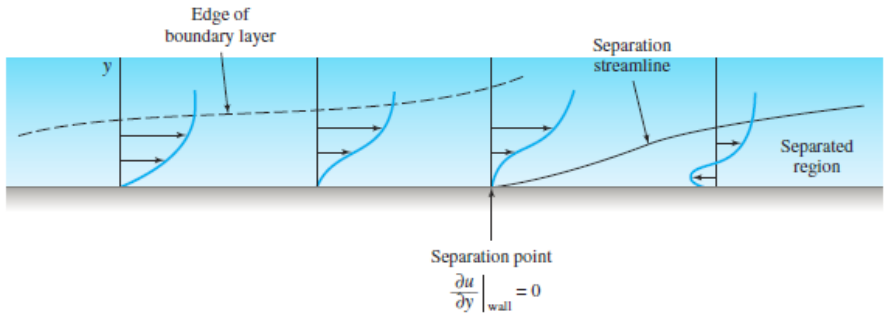


Figure 8.6 Flow separation on a flat surface due to an adverse pressure gradient.

- Separation is influenced by:
  - Geometry
  - Pressure gradient
  - Reynolds number
  - Wall roughness
  - Free-stream fluctuation intensity (intensity of the disturbances that exist away from the boundary)
  - Wall temperature

- ***The last three have less but sometimes significant influences.***

8.3 Flow around Immersed Bodies

8.3.1 Drag Coefficients

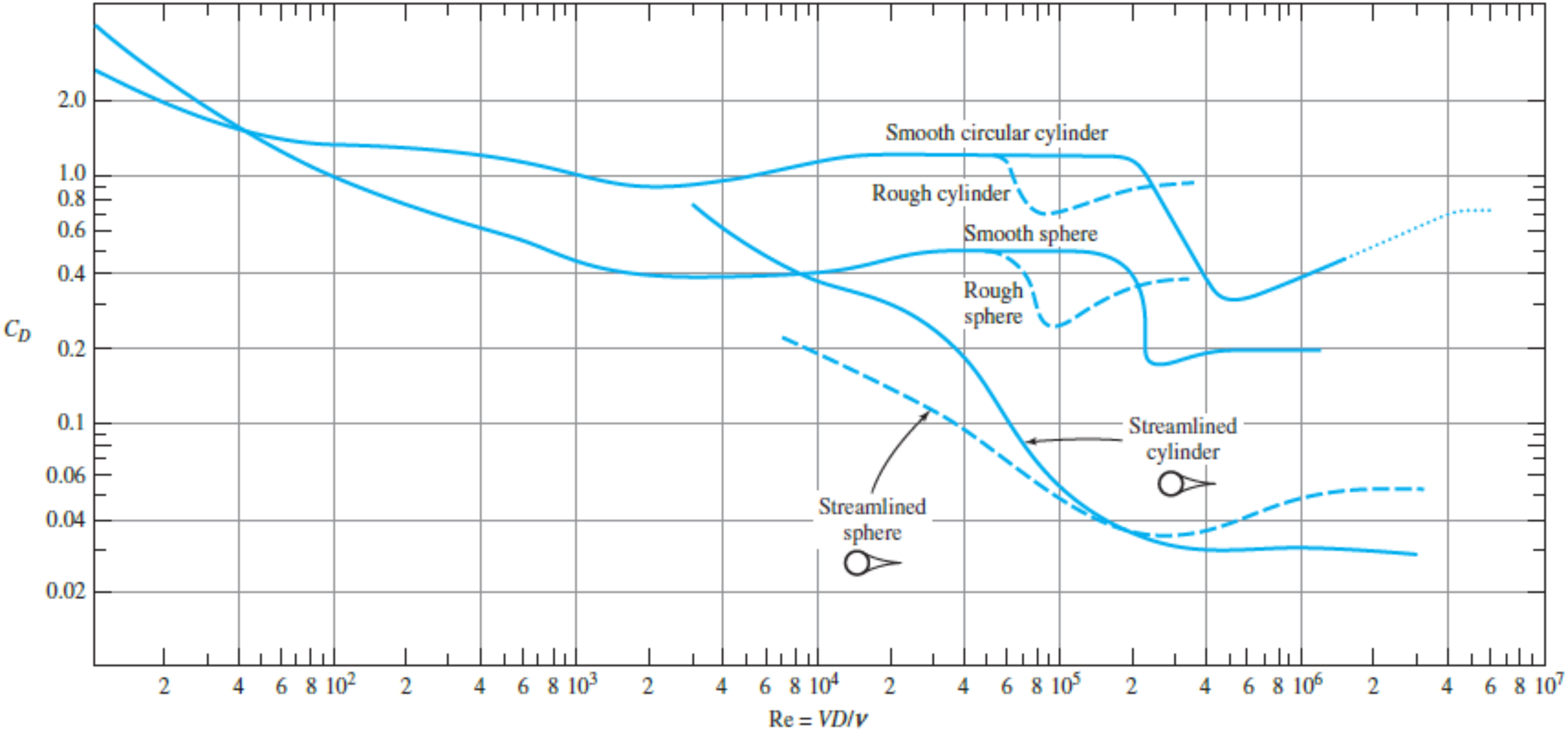
- The primary flow parameter that influences flow is the Reynolds number.
  - Neglecting gravity, thermal, and surface tension effects.

- For a sphere, at Re < 1, there is creeping flow with no separation.

$$C_D = \frac{24}{\mathrm{Re}} \qquad \mathrm{Re} < 1$$

8.3 Flow around Immersed Bodies

8.3.1 Drag Coefficients



Drag coefficients for flow around a long cylinder and a sphere

8.3 Flow around Immersed Bodies

8.3.1 Drag Coefficients

Table 8.1 Drag Coefficients of Finite-Length Circular Cylinders[a] with Free Ends[b] and of Infinite-Length Elliptic Cylinders

Circular cylinder		*Elliptic cylinder*[c]		
$\frac{Length}{Diameter}$	$\frac{C_D}{C_{D\infty}}$	$\frac{Major\ axis}{Minor\ axis}$	Re	C_D
∞	1	2	4×10^4	0.6
40	0.82	4	10^5	0.46
20	0.76	4	2.5×10^4 to 10^5	0.32
10	0.68	8	2.5×10^4	0.29
5	0.62	8	2×10^5	0.20
3	0.62			
2	0.57			
1	0.53			

[a] $C_{D\infty}$ is the drag coefficient for the infinite-length circular cylinder obtained in Figure 8.9.
[b] If one end is fixed to a solid surface, double the length of the cylinder.
[c] Flow is in the direction of the major axis.

8.3 Flow around Immersed Bodies

8.3.1 Drag Coefficients

Object		Re	C_D
Square cylinder $\rightarrow$	$L/w = \infty$	$> 10^4$	2.0
	$L/w = 1$ (cube)	$> 10^4$	1.1
rounded corners ($r = 0.2w$)	$L/w = \infty$	10^5	1.2
Rectangular plates $\rightarrow \vert w$	$L/w = \infty$	$> 10^3$	2.0
	20	$> 10^3$	1.5
	5	$> 10^3$	1.2
	1	$> 10^3$	1.1
Circular cylinder $\rightarrow$	$L/D = 0.1$ (disk)	$> 10^3$	1.1
	4	$> 10^3$	0.9
	7	$> 10^3$	1.0
Semicircular cylinder $\rightarrow$ D		$> 10^4$	2.2
$\rightarrow$ (		$> 10^4$	1.2
Semicircular shell $\rightarrow$)		2×10^4	2.3
$\rightarrow$ (		2×10^4	1.1
Equilateral cylinders $\rightarrow \triangleleft$	2.0	$> 10^4$	2.0
$\rightarrow \triangleright$	1	$> 10^4$	1.4

Drag Coefficients for various blunt objects

8.3 Flow around Immersed Bodies

8.3.1 Drag Coefficients (contd)

Object		Re	C_D
Cone →	$\alpha = 30^\circ$	$> 10^4$	0.6
	$\alpha = 60^\circ$	$> 10^4$	0.8
	$\alpha = 90^\circ$	$> 10^4$	1.2
Solid hemisphere →		$> 10^4$	1.2
→		$> 10^4$	0.4
Hollow hemisphere →		$> 10^4$	1.4
→		$> 10^4$	0.4
Parachute		$> 10^7$	1.4
Automobile			
1920	—	$> 10^5$	0.80
Modern, with square corners	—	$> 10^5$	0.30
Modern, with rounded corners	—	$> 10^5$	0.29
Van		$> 10^5$	0.42
Bicycle, upright rider			1.1
racing, bent over			0.9
racing, drafting			0.5
Semitruck, standard			0.96
with streamlined deflector			0.76
with deflector and gap seal			0.70

Drag Coefficients for various blunt objects

8.3 Flow around Immersed Bodies

A square sign, 3 m × 3 m, is attached to the top of a 18-m-high pole which is 30 cm in diameter (Figure E8.1). Approximate the maximum moment that must be resisted by the base for a wind speed of 30 m/s.

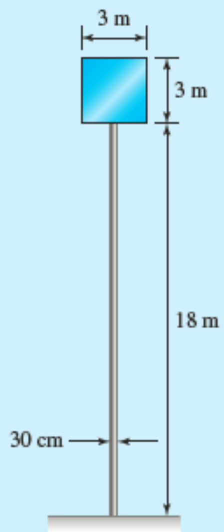


Figure E8.1

Solution

The maximum force F_1 acting on the sign occurs when the wind is normal to the sign; it is

$$F_1 = C_D \times \frac{1}{2}\rho V^2 A$$

$$= 1.1 \times \frac{1}{2} \times 1.2 \times 30^2 \times 3^2 = 5346\ \text{N}$$

8.3 Flow around Immersed Bodies

where C_D is found in Table 8.2 and we use the standard value $\rho = 1.2\ \text{kg/m}^3$ since it was not given. The force F_2 acting on the cylindrical pole is (using the projected area as $A = 18 \times 0.3\ \text{m}^2$)

$$F_2 = C_D \times \frac{1}{2}\rho V^2 A$$
$$= 0.8 \times \frac{1}{2} \times 1.2 \times 30^2 \times 5.4 = 2332.8\ \text{N}$$

where C_D is found from Figure 8.9 with $\text{Re} = 30 \times 0.3/1.46 \times 10^{-5} = 6.2 \times 10^5$, assuming a high-intensity fluctuation level (i.e., a rough cylinder); since neither end is free, we do not use the multiplication factor of Table 8.1.

The resisting moment that must be supplied by the supporting base is

$$M = d_1F_1 + d_2F_2$$
$$= 19.5 \times 5346 + 9 \times 2332.8 = \underline{125\ \text{kN}\cdot\text{m}}$$

assuming that the forces act at the centers of their respective areas.

8.3 Flow around Immersed Bodies

Determine the terminal velocity of a 300-mm-diameter smooth sphere ($S = 1.02$) if it is released from rest in (a) air at 20°C and (b) water at 20°C.

Solution

(a) When terminal velocity is reached by a falling object, the weight of the object is balanced by the drag force acting on the object. Using $\Sigma F = 0$ and Eq. 8.1.1, we have

$$W = F_D$$

$$\therefore \gamma_{\text{sphere}} \times \frac{4}{3}\pi R^3 = C_D \times \frac{1}{2}\rho V^2 A$$

Using $\gamma_{\text{sphere}} = S\gamma_{\text{water}}$ and projected area $A = \pi R^2$, this becomes

$$S\gamma_{\text{water}} \times \frac{4}{3}\pi R^3 = C_D \times \frac{1}{2}\rho V^2 \pi R^2$$

The velocity can now be expressed as

$$V = \left(\frac{8RS\gamma_{\text{water}}}{3\rho C_D}\right)^{1/2} = \left(\frac{8 \times 0.15 \text{ m} \times 1.02 \times 9800 \text{ N/m}^3}{3 \times 1.20 \text{ kg/m}^3 \times C_D}\right)^{1/2} = \frac{57.7}{\sqrt{C_D}}$$

The Reynolds number should be quite large so $C_D = 0.2$ from Figure 8.9. Then

$$V = \frac{57.7}{\sqrt{0.2}} = \underline{129 \text{ m/s}}$$

8.3 Flow around Immersed Bodies

We must check the Reynolds number to verify the C_D value assumed. It is

$$\mathrm{Re} = \frac{VD}{\nu} = \frac{129 \times 0.3}{1.6 \times 10^{-5}} = 2.42 \times 10^{6}$$

This is beyond the end of the curve where data are unavailable; we will assume that the drag coefficient is unchanged at 0.2, so the terminal velocity is 129 m/s.

(b) For the sphere falling in water, we must include the buoyancy force B acting in the same direction as the drag force F_D. Hence the summation of forces yields

$$W = F_D + B$$

$$\therefore \gamma_{\text{sphere}} \times \frac{4}{3}\pi R^3 = C_D \times \frac{1}{2}\rho V^2 A + \gamma_{\text{water}} \times \frac{4}{3}\pi R^3$$

8.3 Flow around Immersed Bodies

This gives

$$(S-1)\gamma_{\text{water}} \times \frac{4}{3}\pi R^3 = C_D \times \frac{1}{2}\rho V^2 \pi R^2$$

Using $\rho = 1000\ \text{kg/m}^3$, there results

$$V = \left(\frac{8R(S-1)\gamma_{\text{water}}}{3\rho C_D}\right)^{1/2} = \left(\frac{8 \times 0.15 \times 0.02 \times 9800}{3 \times 1000 \times C_D}\right)^{1/2} = \frac{0.28}{\sqrt{C_D}}$$

We anticipate the Reynolds number being lower than in part (a), so let's assume that it is in the range $2 \times 10^4 < \text{Re} < 2 \times 10^5$. Then $C_D = 0.5$ and there results

$$V = \underline{\underline{0.40\ \text{m/s}}}$$

This gives a Reynolds number of

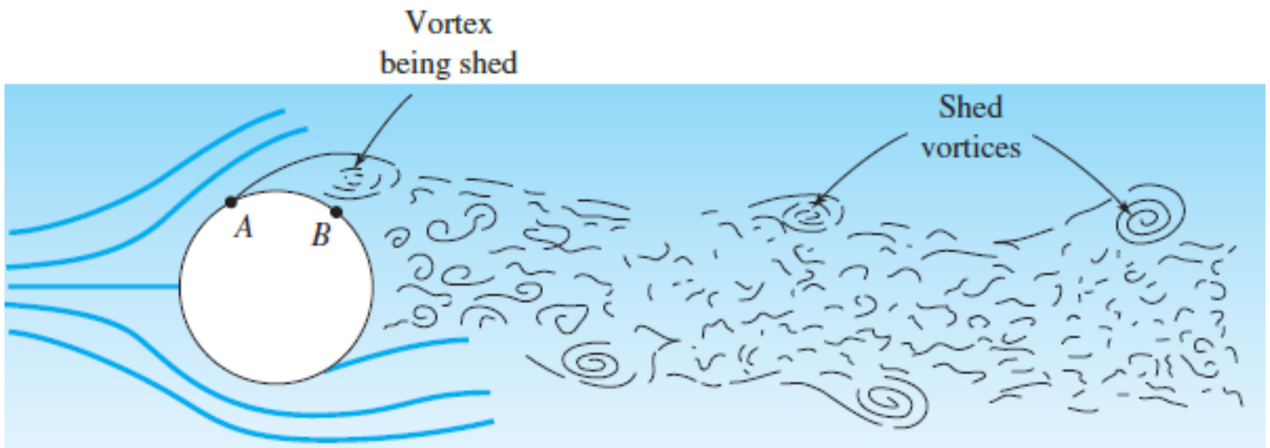
$$\text{Re} = \frac{VD}{\nu} = \frac{0.40 \times 0.3}{10^{-6}} = 1.2 \times 10^5$$

This is in the required range, so the terminal velocity is expected to be 0.40 m/s. Of course, if the sphere were roughened (sand glued to the surface), the C_D value would be less and the velocity would be greater.

8.3 Flow around Immersed Bodies

8.3.2 Vortex Shedding

- When long, blunt objects (such as circular cylinders) are placed normal to a fluid flow:
  - Vortices/eddies are shed regularly and alternately from opposite sides.
  - The flow downstream is called a **Kármán vortex street**. (40 < Re < 10000, with turbulence above Re = 300)



8.3 Flow around Immersed Bodies

8.3.2 Vortex Shedding

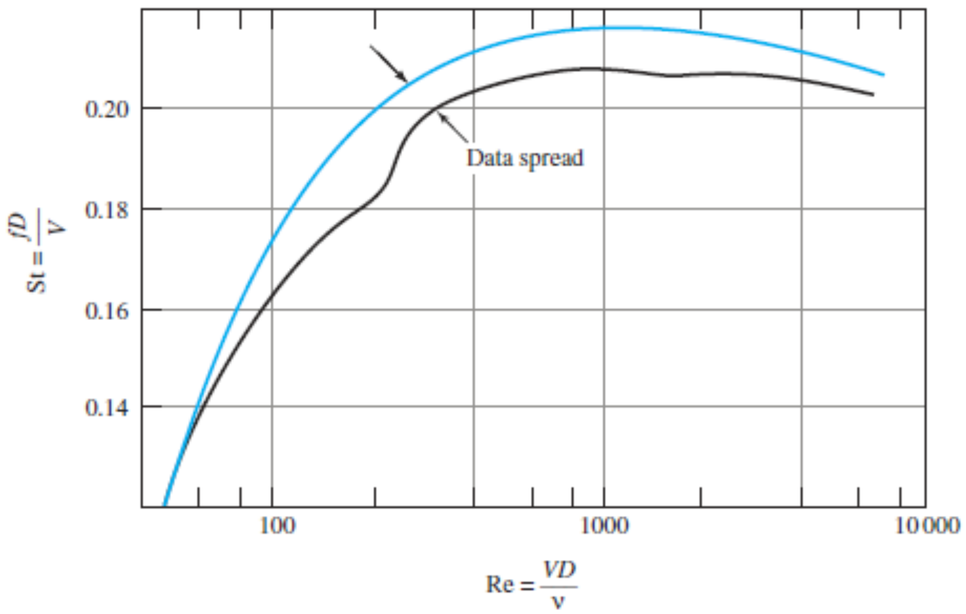
- Dimensional analysis is applied to find this shedding frequency.
  - For high Re (negligible viscous effects), shedding frequency depends on velocity and diameter $f = f(V, D)$.
  - This is the dimensionless Strouhal number.

$$\mathrm{St} = \frac{fD}{V}$$

- The Strouhal number is constant (0.21) over 300 < Re < 10000.
  - Frequency is directly proportional to the velocity over a large Re range.

8.3 Flow around Immersed Bodies

8.3.2 Vortex Shedding



- Vortex shedding is considered when designing towers/bridges.
  - When a vortex is shed, a small force is applied to the structure.
  - If the shedding frequency is close to the natural frequency → **Resonance**

8.3 Flow around Immersed Bodies

The velocity of a slow-moving, 30°C air stream is to be measured using a cylinder and a pressure tap located between points *A* and *B* on the cylinder in Figure 8.10a. The velocity range is expected to be $0.1 < V < 1$ m/s. What size cylinder should be selected and what frequency would be observed by the pressure-measuring device for $V = 1$ m/s?

Solution

The Reynolds number should be in the vortex shedding range, say 4000. For the maximum velocity the diameter would be found as follows:

$$4000 = \frac{VD}{\nu}$$

$$= \frac{1.0 \text{ m/s} \times D}{1.6 \times 10^{-5} \text{ m}^2/\text{s}}$$

$$\therefore D = 0.064 \text{ m} \quad \text{Select } D = \underline{60 \text{ mm}}$$

At $V = 0.1$ m/s the Reynolds number is $0.1 \times 0.06/1.6 \times 10^{-5} = 375$. Vortex shedding would occur, so this is acceptable. The expected vortex shedding frequency at $V = 1.0$ m/s is found using a Strouhal number from Figure 8.10b of 0.21. Hence

$$0.21 = \frac{fD}{V}$$

$$= \frac{f \times 0.06}{1.0}$$

$$\therefore f = \underline{3.5 \text{ hertz}}$$

8.3 Flow around Immersed Bodies

8.3.3 Streamlining

- If a flow must remain attached to the surface of a blunt object (cylinder/ sphere):
    - It must move into areas of higher pressure as it goes into the rear stagnation point.
    - At high Reynolds numbers (Re > 10), the slow-moving boundary layer flow cannot make it to the high-pressure area near the rear stagnation point (separation occurs).
- Streamlining: Reduces the high pressure at the rear of the object so the slow-moving surface flow can move rearward (into a slightly high-pressure region).
    - Separation region will be reduced to a small percentage of the initial separated region on the blunt object.
    - For effective streamlining trailing edge angle < 20 degrees.

8.3 Flow around Immersed Bodies

8.3.3 Streamlining

- For a streamlined body:
  - Surface area is greatly increased.
  - Reduces most of the pressure drag, but increases the shear drag on the surface.
  - Usually eliminates periodic shedding of vortices.

- The streamlined body cannot be so long that the shear drag is larger than the pressure drag plus shear drag for a short body.
  - *Needs to be optimized.*

8.3 Flow around Immersed Bodies

A strut on a stunt plane traveling at 60 m/s is 40 mm in diameter and 240 mm long. Calculate the drag force acting on the strut as a circular cylinder, and as a streamlined strut, as shown in Figure E8.4. Neglect any viscous drag. Would you expect vortex shedding from the circular cylinder?

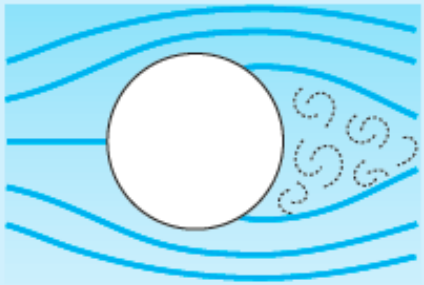


Figure E8.4(a)

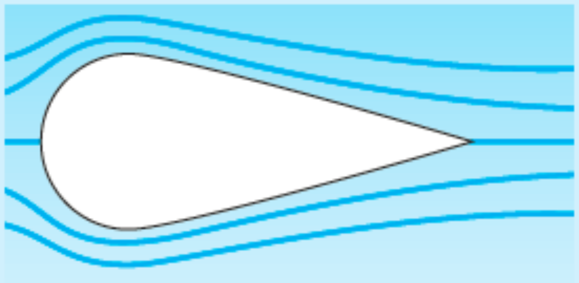


Figure E8.4(b)

Solution

The Reynolds number associated with the cylinder and the streamlined strut is, assuming air at $T = 20°C$,

$$\mathrm{Re} = \frac{VD}{\nu}$$

$$= \frac{60 \times 0.04}{1.5 \times 10^{-5}} = 1.6 \times 10^{5}$$

8.3 Flow around Immersed Bodies

Assuming a smooth surface as in Figure E8.4a, the drag coefficient is $C_D = 1.2$ from Figure 8.9. The drag force is then

$$F_D = C_D \times \frac{1}{2}\rho V^2 A$$

$$= 1.2 \times \frac{1}{2} \times 1.20\,\text{kg/m}^3 \times 60^2\,\text{m}^2/\text{s}^2 \times (0.24 \times 0.04)\,\text{m}^2 = \underline{24.9\text{ N}}$$

For the streamlined strut of Figure E8.4b, Figure 8.9 yields $C_D = 0.04$. The drag force is

$$F_D = C_D \times \frac{1}{2}\rho V^2 A$$

$$= 0.04 \times \frac{1}{2} \times 1.20 \times 60^2 \times (0.24 \times 0.04) = \underline{0.82\text{ N}}$$

This is a reduction of 97% in the drag, a rather substantial reduction.

Vortex shedding is not to be expected on the circular cylinder; the Reynolds number is too high. (See Figure 8.10.)

8.3 Flow around Immersed Bodies

8.3.4 Cavitation

- This is the rapid change of phase from liquid to vapor that occurs when the local pressure is equal to or less than the vapor pressure.

8.3 Flow around Immersed Bodies

8.3.4 Cavitation

- There are four types of cavitation:
  1. Traveling cavitation: *Exists when vapor bubbles/cavities are formed, are swept downstream, and then collapse.*
  2. Fixed cavitation: *Exists when a fixed cavity of vapor exists as a separated region.*
  3. Vortex cavitation: *Found in the high-velocity (low pressure) core of a vortex (e.g., tip vortex leaving a propeller).*
  4. Vibratory cavitation: *Exists when a pressure wave moves in a liquid (pressure pulse of high pressure followed by a low pressure).*
     - The low-pressure part of the wave causes cavitation.
- The first type of cavitation causes a lot of damage.
  - Instantaneous pressures caused from the collapse are very high (~1400 MPa).

8.3 Flow around Immersed Bodies

8.3.4 Cavitation

- Cavitation occurs when the cavitation number (σ) is less than the *critical cavitation number* (σ_{crit}).
  - Depends on the body geometry and Reynolds number.

$$\sigma = \frac{p_\infty - p_v}{\frac{1}{2}\rho V^2}$$

p_∞: Absolute pressure of the free stream
p_v: Vapor pressure

- As σ keeps decreasing, the cavitation increases in intensity (moving from traveling cavitation to fixed cavitation to supercavitation).

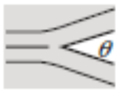
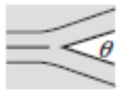
8.3 Flow around Immersed Bodies

8.3.4 Cavitation

$$\sigma = \frac{p_\infty - p_v}{\frac{1}{2}\rho V^2}$$

- The drag coefficient for small numbers is: $C_D(\sigma) = C_D(0)(1+\sigma)$

Table 8.3 Drag Coefficients for Zero Cavitation Number for Blunt Objects for $\mathrm{Re} \cong 10^5$

Two-dimensional body				*Axisymmetric body*		
Geometry		θ	$C_D(0)$	*Geometry*	θ	$C_D(0)$
Flat plate		—	0.88	Disk	—	0.8
Circular cylinder		—	0.50	Sphere	—	0.30
		120	0.74	Cone	120	0.64
Wedge		90	0.64		90	0.52
		60	0.49		60	0.38
		30	0.28		30	0.20

8.3 Flow around Immersed Bodies

8.3.4 Cavitation

- The hydrofoil (airfoil-type body) is a shape that is associated with cavitation.
  - Used to lift bodies out of water.

Table 8.4 Drag and Lift Coefficients and Critical Cavitation Number for a Typical Hydrofoil

Angle (°)	*Lift coefficient* C_L	*Drag coefficient* C_D	*Critical cavitation number* σ_{crit}
−2	0.2	0.014	0.5
0	0.4	0.014	0.6
2	0.6	0.015	0.7
4	0.8	0.018	0.8
6	0.95	0.022	1.2
8	1.10	0.03	1.8
10	1.22	0.04	2.5

8.3 Flow around Immersed Bodies

A hydrofoil is to operate 50 cm below the surface of 15°C water at an angle of attack of 8° and travel at 14 m/s. If its chord length is 60 cm and it is 1.8 m long, calculate its lift and drag. Is cavitation present?

Solution

The absolute pressure p_∞ is

$$p_\infty = \gamma h + p_{atm}$$
$$= 9810 \times 0.5 + 1.01 \times 10^5 = 105.9 \text{ kPa absolute}$$

The vapor pressure is $p_v = 1.765$ kPa, so

$$\sigma = \frac{p_\infty - p_v}{\frac{1}{2}\rho V^2}$$
$$= \frac{(105.9 - 1.765) \times 10^3}{\frac{1}{2} \times 1000 \times 14^2} = 1.06$$

Answering the last question first, we see that this is less than 1.8; hence cavitation exists.

The lift force is, finding C_L in Table 8.4,

$$F_L = C_L \times \frac{1}{2}\rho V^2 A$$
$$= 1.1 \times \frac{1}{2} \times 1000 \text{ kg/m}^3 \times 14^2 \text{ m}^2/\text{s}^2 \times (0.6 \times 1.8) \text{ m}^2 = \underline{116.4 \text{ kN}}$$

The drag force is, finding C_D in Table 8.4,

$$F_D = C_D \times \frac{1}{2}\rho V^2 A$$
$$= 0.03 \times \frac{1}{2} \times 1000 \times 14^2 (0.6 \times 1.8) = \underline{3180 \text{ N}}$$

8.3 Flow around Immersed Bodies

8.3.5 Added Mass

- If a body accelerates in a fluid, some of the fluid surrounding the body also accelerates.
  - This can be accounted for by adding a small mass m_a, to the mass of the body.
  - Hence, summing forces for a symmetrical body:

$$F - F_D = (m + m_a)\frac{dV_B}{dt}$$

- The added mass can be related to the fluid mass (m_f) by:

$$m_a = km_f$$

k: Added mass coefficient
Sphere k = 0.5
Ellipsoid k = 0.2
Long cylinder k = 1.0

8.3 Flow around Immersed Bodies

A sphere with specific gravity 2.5 is released from rest in water. Calculate its initial acceleration. What is the percentage error if the added mass is ignored?

Solution

The summation of forces in the vertical direction, with zero drag, is

$$W - B = (m + m_a)\frac{dV_B}{dt}$$

where B is the buoyant force. Substituting in the appropriate quantities gives, letting $\forall$ = sphere volume,

$$S\gamma_{\text{water}}\forall - \gamma_{\text{water}}\forall = (\rho_{\text{water}}S\forall + 0.5\rho_{\text{water}}\forall)\frac{dV_B}{dt}$$

This gives

$$g(S-1) = (S + 0.5)\frac{dV_B}{dt}$$

Hence

$$\frac{dV_B}{dt} = \frac{g(S-1)}{S+0.5} = \frac{9.8(2.5-1)}{2.5+0.5} = \underline{4.90\ \text{m/s}^2}$$

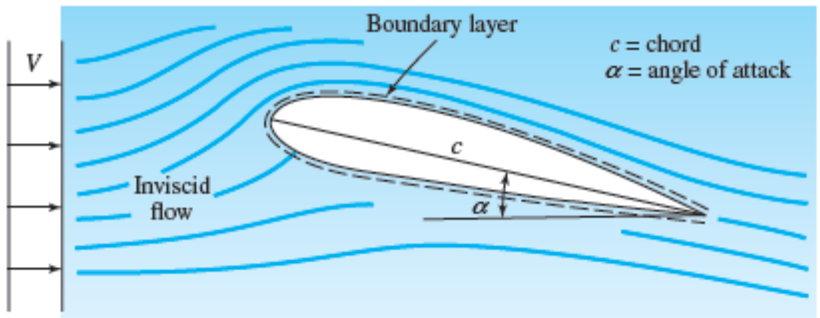
If we ignored the added mass, the acceleration would be

$$\frac{dV_B}{dt} = \frac{g(S-1)}{S} = \frac{9.8(2.5-1)}{2.5} = 5.88\ \text{m/s}^2$$

This is an error of 20%.

8.4 Lift and Drag on Airfoils

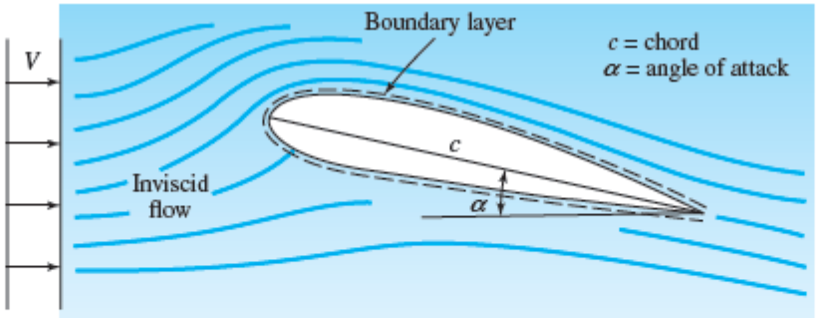
- An airfoil is a streamlined body that reduces adverse pressure gradient.
  - Hence separation doesn't occur (at small angles of attack).
  - Drag is mainly due to wall shear stress (viscous effects in the boundary layer).



8.4 Lift and Drag on Airfoils

- For this thin boundary layer, lift can be approximated by integrating the pressure distribution.
- Drag is found by solving the boundary layer equations (Navier-Stokes equations) for shear stress:
    - Then integrate.

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 cL} \qquad C_L = \frac{F_L}{\frac{1}{2}\rho V^2 cL}$$



8.4 Lift and Drag on Airfoils

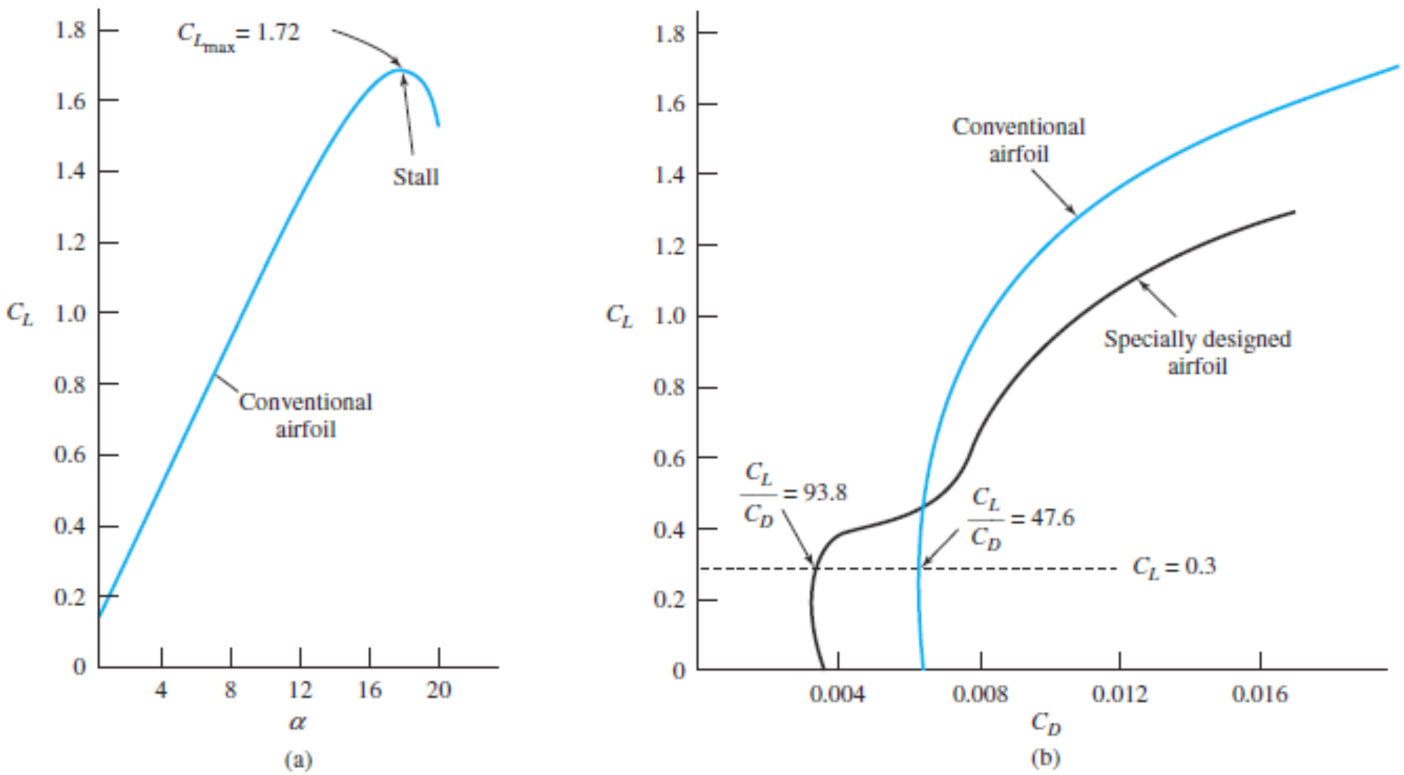


Figure 8.13 Lift and drag coefficients for airfoils with $Re = Vc/\nu \simeq 9 \times 10^6$ (c is the chord length).

8.4 Lift and Drag on Airfoils

- Maximum lift coefficients can be around 1.5
- Minimum drag coefficients for some particular airfoils can be as low as 0.0035

- **The cruise condition (design lift coefficient) is around 0.3, which is near the minimum drag coefficient condition.**
  - Around an angle of attack of 2° (maximum stall condition of 16°)

- Conventional airfoils aren't symmetric.
  - Will have a positive lift coefficient at zero angle of attack.

- **Lift is proportional to AOA, but deviates from a linear function just before stall.**
- **Drag is proportional to AOA (upto 5°), then changes in a nonlinear relation.**

8.4 Lift and Drag on Airfoils

- The total lift on an aircraft is supplied primarily by the airfoil.
  - The effective length of the airfoil in this calculation is taken from the tip-to-tip distance.
  - This is the **wingspan.**

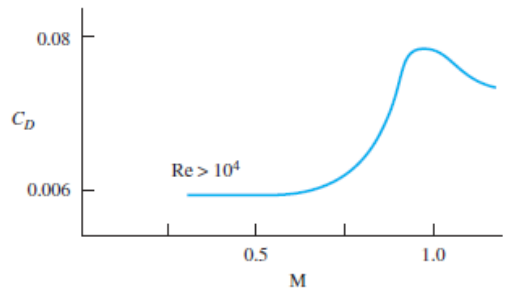


Figure 8.15 Drag coefficient as a function of Mach number (speed) for a typical unswept airfoil.

- The drag coefficient is constant until around M = 0.75
- Then there is an increase until the Mach number reaches unity.
- After this, the drag coefficient drops.
- Hence, to avoid high drag coefficients M = 1 should be avoided.
  - At M = 1, there are regions of flow that oscillate from subsonic to supersonic. These forces need to be avoided.

8.6 Boundary-Layer Theory

8.6.1 General Background

- For high Re flows:
  - Viscous effects are confined to the thin layer of fluid (boundary layer) next to the body and to the wake downstream of the body.
- The edge of the boundary layer is arbitrarily defined as:
  - The locus of points where the velocity equals 99% of the free-stream velocity.

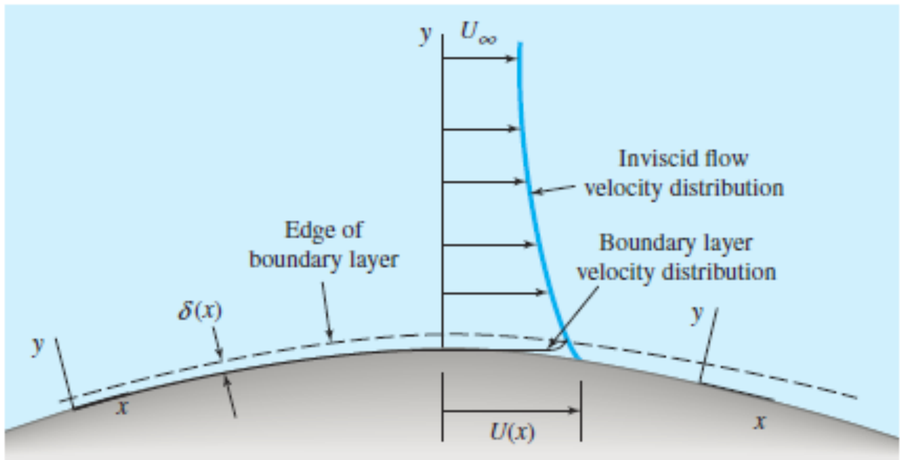


Figure 8.21 Boundary layer on a curved surface.

For a thin boundary layer, the pressure in the boundary layer is the pressure at the wall [inviscid flow solution].

8.6 Boundary-Layer Theory

8.6.1 General Background

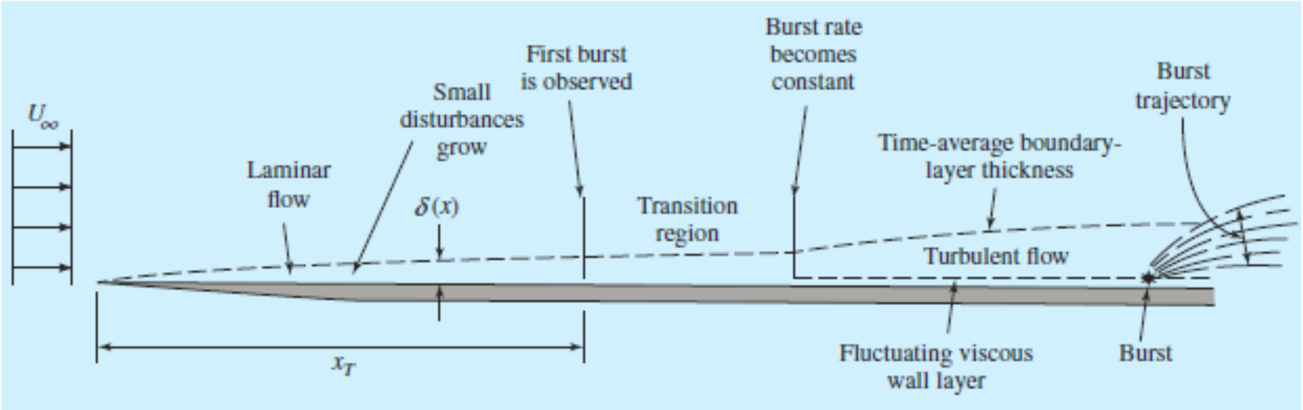


Figure 8.22 Boundary layer with transition.

- The boundary layer begins as a laminar flow (with zero thickness) at the leading edge of a flat plate.
- After a distance x_T, laminar-to-turbulent transition. x_T depends on:
  - Free-stream velocity
  - Viscosity
  - Pressure gradient
  - Wall roughness
  - Free-stream fluctuation level
  - Wall rigidity

8.6 Boundary-Layer Theory

8.6.1 General Background

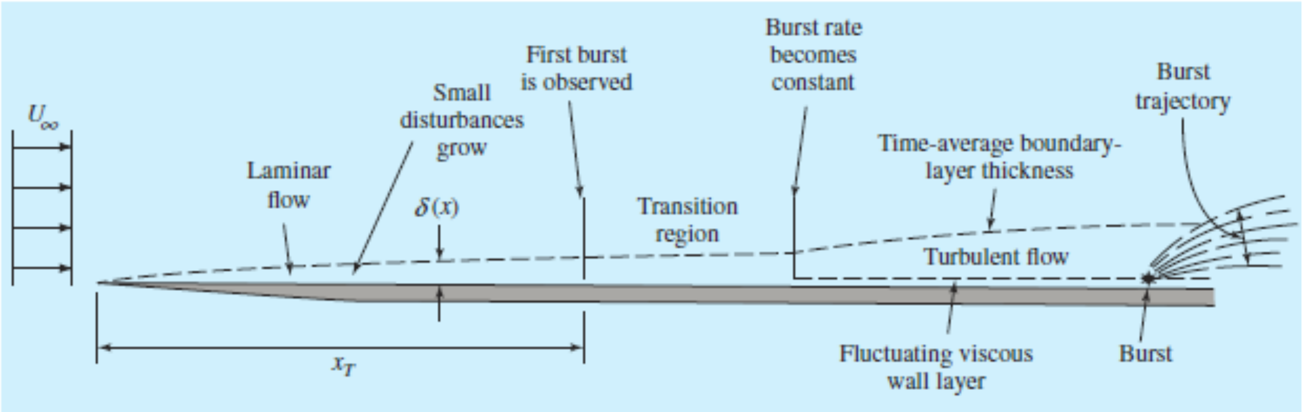


Figure 8.22 Boundary layer with transition.

- For a flat plate with zero pressure gradient, the transition occurs:
  - Flow on rough plates or with high free-stream fluctuation intensity →
  - $U_\infty x_T/v = 3 \times 10^5$
  - Flow on smooth rigid plates with low free-stream fluctuation intensity →
  - $U_\infty x_T/v = 5 \times 10^5$
- The quantity $[U_\infty x/v]$ is the **local Reynolds number**.
  - Hence $U_\infty x_T/v$ is the **critical Reynolds number**.

8.6 Boundary-Layer Theory

8.6.1 General Background

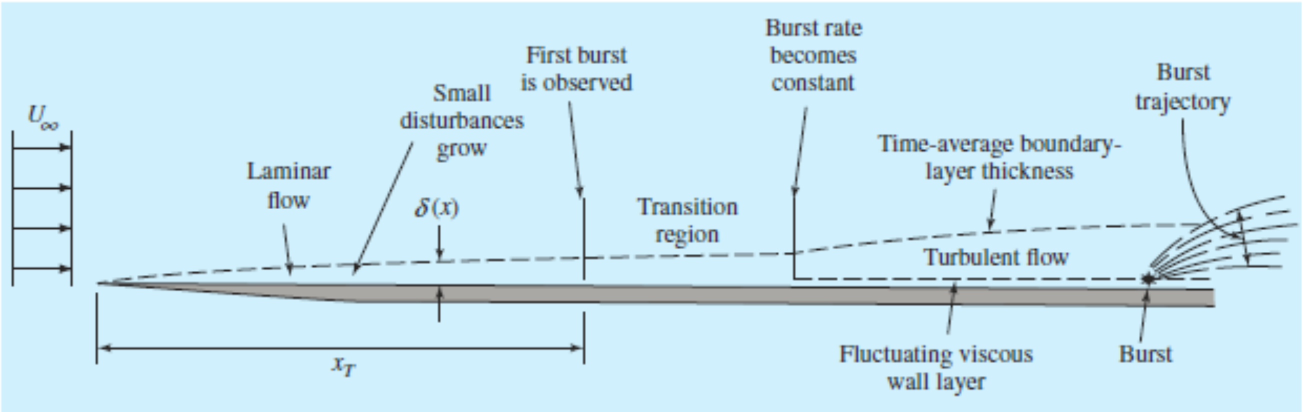


Figure 8.22 Boundary layer with transition.

- Assume flow is laminar up to x_T and turbulent afterwards.
- The turbulent boundary layer thickens much more rapidly than the laminar layer.
    - Has greater wall shear and a greater slope at the wall.

8.6 Boundary-Layer Theory

8.6.2 Von Kármán Integral Equation

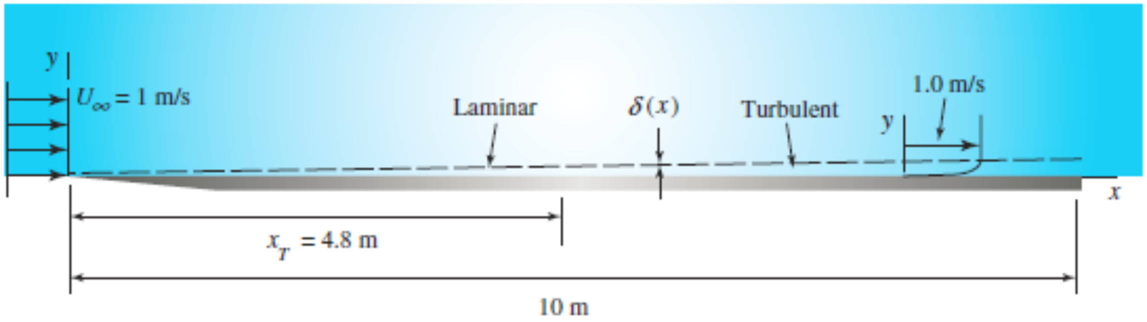


Figure 8.24 Boundary layer in air with $Re_{crit} = 3 \times 10^5$ (approximately to scale).

- As seen from the velocity profile:
  - Velocity goes from $u = 0.99U_\infty$ at $y = \delta$ to $u = 0$ at $y = 0$ over a small distance (boundary layer thickness).
  - Hence the velocity profile for laminar and turbulent flow can be predicted well.

8.6 Boundary-Layer Theory

8.6.2 Von Kármán Integral Equation

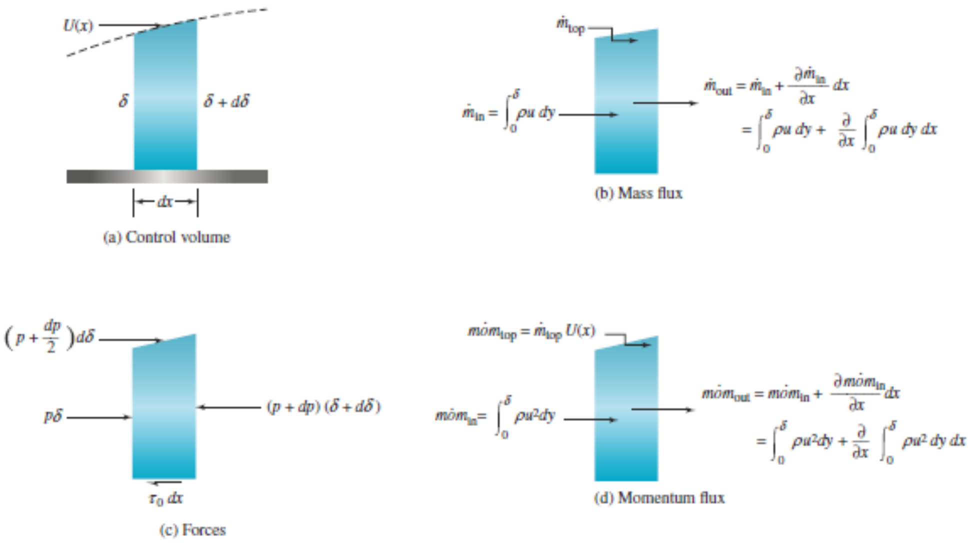


Figure 8.25 Control volume for a boundary layer with variable $U(x)$.

- For an infinitesimal control volume, the integral continuity equation is:

$$\dot{m}_{top} = \dot{m}_{out} - \dot{m}_{in} = \frac{\partial}{\partial x} \int_0^\delta \rho u\, dy\, dx$$

- The integral momentum equation then becomes:

$$\sum F_x = \dot{mom}_{out} - \dot{mom}_{in} - \dot{mom}_{top}$$

- Hence, from momentum flux, the von Kármán integral equation is:

$$\tau_0 + \delta \frac{dp}{dx} = U(x) \frac{d}{dx} \int_0^\delta \rho u\, dy - \frac{d}{dx} \int_0^\delta \rho u^2\, dy$$

8.6 Boundary-Layer Theory

8.6.2 Von Kármán Integral Equation

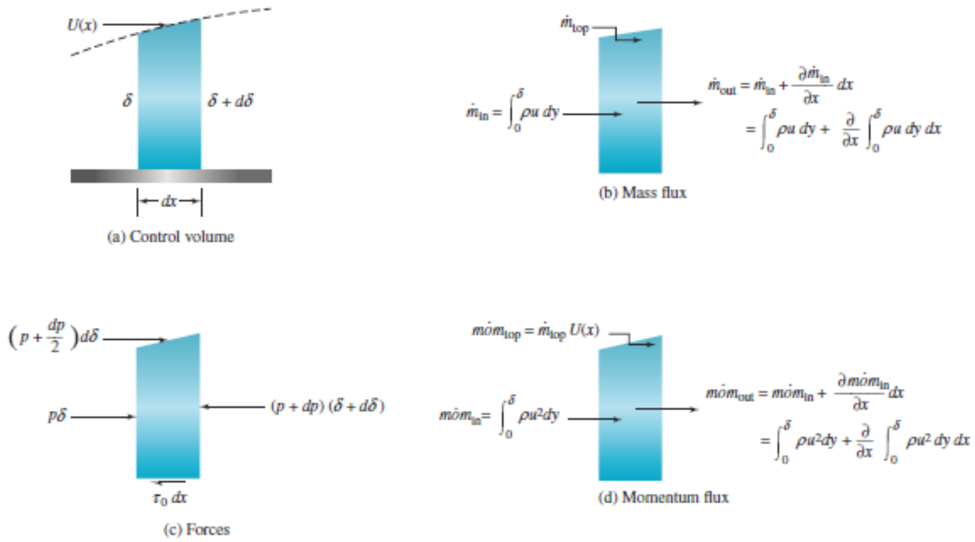


Figure 8.25 Control volume for a boundary layer with variable $U(x)$.

- For flow over a flat plate with zero pressure gradient, the simplified equation is:

$$\tau_0 = \frac{d}{dx}\int_0^{\delta} \rho u U_\infty \, dy - \frac{d}{dx}\int_0^{\delta} \rho u^2 \, dy = \frac{d}{dx}\int_0^{\delta} \rho u (U_\infty - u)\, dy$$

Constant density

$$\tau_0 = \rho U_\infty^2 \frac{d\theta}{dx}$$

8.6 Boundary-Layer Theory

8.6.2 Von Kármán Integral Equation

- ***Displacement thickness δ_d***: The displacement of the streamlines in the free stream as a result of velocity deficits in the boundary layer.
- ***Momentum thickness θ***: The equivalent thickness of a fluid layer (velocity *U*) with momentum equal to the momentum lost from friction.
  - Used as a characteristic length in turbulent boundary-layer analysis.

$$\delta_d = \frac{1}{U}\int_0^\delta (U - u)\ dy$$

$$\theta = \frac{1}{U^2}\int_0^\delta u(U - u)\,dy$$

8.6 Boundary-Layer Theory

8.6.3 Approximate Solution to the Laminar Boundary Layer

- The von Kármán integral equation can be used to obtain an approximation to the laminar boundary layer on a flat plate with zero pressure gradient.

- Propose a velocity profile with the below conditions.
  - From the velocity profile sketch and the x-component of the Navier-Stokes equation.

$$\begin{aligned} u &= 0 && \text{at} \quad y = 0 \\ u &= U_\infty && \text{at} \quad y = \delta \\ \frac{\partial u}{\partial y} &= 0 && \text{at} \quad y = \delta \\ \frac{\partial^2 u}{\partial y^2} &= 0 && \text{at} \quad y = 0 \end{aligned}$$

The velocity profile for a laminar flow is then:

$$\frac{u}{U_\infty} = \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

8.6 Boundary-Layer Theory

8.6.3 Approximate Solution to the Laminar Boundary Layer

- From the velocity profile in a laminar flow: $\frac{u}{U_\infty} = \frac{3}{2}\frac{y}{\delta} + \frac{1}{2}\left(\frac{y}{\delta}\right)^3$

- $\delta(x)$ and $\tau_0(x)$ are found to be as follows from von Kármán's integral equation:

$$\tau_0 = \mu\left(U_\infty \frac{3}{2\delta}\right)$$

$$\tau_0 = 0.323\rho U_\infty^2 \sqrt{\frac{\nu}{xU_\infty}} = \frac{0.323\rho U_\infty^2}{\sqrt{\mathrm{Re}_x}}$$

$$\delta = 4.65\sqrt{\frac{\nu x}{U_\infty}} = 4.65\frac{x}{\sqrt{\mathrm{Re}_x}}$$

8.6 Boundary-Layer Theory

8.6.3 Approximate Solution to the Laminar Boundary Layer

- Using the cubic velocity profile and the shearing stress, the **local skin friction coefficient c_f** can be defined:
  - This is a dimensionless wall shearing stress.

$$c_f = \frac{\tau_0}{\frac{1}{2}\rho U_\infty^2}$$

$$= \frac{0.646}{\sqrt{U_\infty x/\nu}} = \frac{0.646}{\sqrt{\mathrm{Re}_x}}$$

Re_L: Reynolds number at the end of the flat plate

$$Re = \frac{U_\infty x}{\nu}$$

- By integrating the wall shear over length L, the **skin friction coefficient C_f** (dimensionless drag force):

$$C_f = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 L}$$

$$= \frac{1.29}{\sqrt{U_\infty L/\nu}} = \frac{1.29}{\sqrt{\mathrm{Re}_L}}$$

8.6 Boundary-Layer Theory

Assume that the velocity profile in a boundary-layer flow can be approximated by a parabolic velocity profile. Calculate the boundary-layer thickness and the wall shear. Compare with those calculated above for the cubic profile.

Solution

The parabolic velocity profile is assumed to be

$$\frac{u}{U_\infty} = A + By + Cy^2$$

The fourth condition, which would be impossible to satisfy, of (8.6.9) is omitted; this leaves

$$\begin{aligned} 0 &= A \\ 1 &= A + B\delta + C\delta^2 \\ 0 &= B + 2C\delta \end{aligned}$$

A simultaneous solution provides

$$A = 0 \qquad B = \frac{2}{\delta} \qquad C = -\frac{1}{\delta^2}$$

The velocity profile is then

$$\frac{u}{U_\infty} = 2\frac{y}{\delta} - \frac{y^2}{\delta^2}$$

This is substituted into von Kármán's integral equation (8.6.5) to obtain

$$\begin{aligned}\tau_0 &= \frac{d}{dx}\int_0^\delta \rho U_\infty^2\left(2\frac{y}{\delta}-\frac{y^2}{\delta^2}\right)\left(1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right)dy \\ &= \frac{2}{15}\rho U_\infty^2\frac{d\delta}{dx}\end{aligned}$$

We also use $\tau_0 = \mu\,\partial u/\partial y\,|_{y=0}$; that is,

$$\tau_0 = \mu U_\infty \frac{2}{\delta}$$

Equating the two expressions above, we obtain

$$\delta\, d\delta = 15\frac{\nu}{U_\infty}dx$$

Using $\delta = 0$ at $x = 0$, this is integrated to

$$\delta = 5.48\sqrt{\frac{\nu x}{U_\infty}}$$

This is 18% higher than the value using the cubic but only 10% higher than the more accurate result of $5\sqrt{\nu x/U_\infty}$.

The wall shear is found to be

$$\begin{aligned}\tau_0 &= \frac{2\mu U_\infty}{\delta} \\ &= 0.365\rho U_\infty^2\sqrt{\frac{\nu}{xU_\infty}}\end{aligned}$$

This is 13% higher than the value using the cubic and 10% higher than the more accurate value of $0.332\,\rho U_\infty^2\sqrt{\nu/xU_\infty}$. Because the boundary layer is so thin, there is little difference between a cubic and a parabola or the actual profile; refer to the profile in Figure 8.24.

8.6 Boundary-Layer Theory

8.6.4 Turbulent Boundary Layer: Power-Law Form

- One of two methods used for turbulent boundary-layer flow involves fitting the data for the velocity profile using a *power-law equation.*

$$\frac{\overline{u}}{U_\infty} = \left(\frac{y}{\delta}\right)^{1/n} \qquad n = \begin{cases} 7 & \mathrm{Re}_x < 10^7 \\ 8 & 10^7 < \mathrm{Re}_x < 10^8 \\ 9 & 10^8 < \mathrm{Re}_x < 10^9 \end{cases}$$

- Following the previous methodology (von Kármán), the local skin friction coefficients and the shear stress relations can be found.

8.6 Boundary-Layer Theory

8.6.4 Turbulent Boundary Layer: Power-Law Form

- Two forms of shear stress relationships:

$$\tau_0 = 0.023\,\rho U_\infty^2 \left(\frac{\nu}{U_\infty \delta}\right)^{1/4}$$

$$\tau_0 = \frac{d}{dx}\int_0^\delta \rho U_\infty^2 \left(\frac{y}{\delta}\right)^{1/7}\left[1 - \left(\frac{y}{\delta}\right)^{1/7}\right]dy = \frac{7}{72}\rho U_\infty^2 \frac{d\delta}{dx}$$

- The local skin friction coefficient can be related to the boundary-layer thickness and local Reynolds numbers as:

$$c_f = 0.046\left(\frac{\nu}{U_\infty \delta}\right)^{1/4}$$

$$c_f = 0.059\,\mathrm{Re}_x^{1/5} \qquad \mathrm{Re}_x < 10^7$$

$$\delta = 0.38x\left(\frac{\nu}{U_\infty x}\right)^{1/5} = 0.38x\,\mathrm{Re}_x^{-1/5} \qquad \mathrm{Re}_x < 10^7$$

Assuming turbulent flow from the leading edge

8.6 Boundary-Layer Theory

8.6.4 Turbulent Boundary Layer: Power-Law Form

- The skin friction coefficient can then be found as:

$$C_f = 0.073\,\mathrm{Re}_L^{-1/5} \qquad \mathrm{Re}_L < 10^7$$

$$C_f = 0.073\,\mathrm{Re}_L^{-1/5} - 1700\,\mathrm{Re}_L^{-1} \qquad \mathrm{Re}_L < 10^7$$

- If L is around $3x_T$ or less, then there is a significant laminar section on the leading edge. Equation is modified as such
- Used for a $Re_{crit} = 5 \times 10^5$

- Finally, the displacement and momentum thicknesses can be found as:

$$\delta_d = 0.048\,x\,\mathrm{Re}_x^{-1/5}$$
$$\theta = 0.037\,x\,\mathrm{Re}_x^{-1/5}$$

8.6 Boundary-Layer Theory

Estimate the boundary-layer thickness at the end of a 4-m-long flat surface if the free-stream velocity is $U_\infty = 5$ m/s. Use atmospheric air at 30°C. Also, predict the drag force if the surface is 5 m wide. (a) Neglect the laminar portion of the flow and (b) account for the laminar portion using $\text{Re}_{\text{crit}} = 5 \times 10^5$.

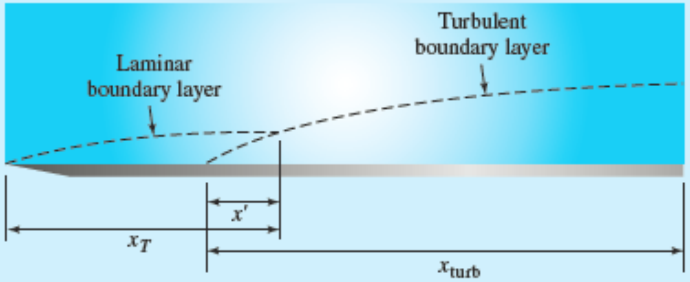


Figure E8.14

Solution

(a) Let us first assume turbulent flow from the leading edge. The boundary-layer thickness is given by Eq. 8.6.27. It is

$$\delta = 0.38x\,\text{Re}_x^{-1/5}$$
$$= 0.38 \times 4 \times \left(\frac{5 \times 4}{1.6 \times 10^{-5}}\right)^{-1/5} = \underline{0.0917 \text{ m}}$$

The drag force is, using Eq. 8.6.29,

$$F_D = C_f \times \frac{1}{2}\rho U_\infty^2 L w$$
$$= 0.073\left(\frac{5 \times 4}{1.6 \times 10^{-5}}\right)^{-1/5} \times \frac{1}{2} \times 1.16\,\text{kg/m}^3 \times 5^2\ \text{m}^2/\text{s}^2 \times 4\,\text{m} \times 5\,\text{m} = \underline{1.28\ \text{N}}$$

8.6 Boundary-Layer Theory

The predictions above assume that $\mathrm{Re}_L < 10^7$. The Reynolds number is

$$\mathrm{Re}_L = \frac{5 \times 4}{1.6 \times 10^{-5}} = 1.25 \times 10^6$$

Hence the calculations are acceptable.

(b) Now let us account for the laminar portion of the boundary layer. Referring to Figure E8.14, the distance x_T is found as follows:

$$\mathrm{Re}_{\mathrm{crit}} = 5 \times 10^5 = \frac{U_\infty x_T}{\nu}$$

$$\therefore x_T = 5 \times 10^5 \times 1.6 \times \frac{10^{-5}}{5} = 1.6\,\mathrm{m}$$

The boundary-layer thickness at x_T is, replacing the constant of 4.65 in Eq. 8.6.16 with the more accurate value of 5,

$$\delta = 5\sqrt{\frac{x\nu}{U_\infty}}$$

$$= 5\sqrt{\frac{1.6\,\mathrm{m} \times 1.6 \times 10^{-5}\,\mathrm{m^2/s}}{5\,\mathrm{m/s}}} = 0.0113\,\mathrm{m}$$

The location of the fictitious origin of the turbulent flow (see Figure E8.14) is found using Eq. 8.6.27 to be

$$x'^{4/5} = \frac{\delta}{0.38}\left(\frac{U_\infty}{\nu}\right)^{1/5}$$

$$\therefore x' = \left(\frac{0.0113}{0.38}\right)^{5/4}\left(\frac{5}{1.6 \times 10^{-5}}\right)^{1/4} = 0.292\,\mathrm{m}$$

8.6 Boundary-Layer Theory

The distance x_{turb} is then $x_{turb} = 4 - 1.6 + 0.292 = 2.69$ m. Using Eq. 8.6.27, the thickness at the end of the surface is

$$\delta = 0.38x\left(\frac{\nu}{U_\infty x}\right)^{1/5}$$

$$= 0.38 \times 2.69 \times \left(\frac{1.6 \times 10^{-5}}{5 \times 2.69}\right)^{1/5} = \underline{0.067\text{ m}}$$

The value of part (a) is 37% too high when compared with this more accurate value.

The more accurate drag force is found using Eq. 8.6.30 to be

$$F_D = C_f \times \frac{1}{2}\rho U_\infty^2 Lw$$

$$= [0.073\,\text{Re}_L^{-1/5} - 1700\,\text{Re}_L^{-1}] \times \frac{1}{2}\rho U_\infty^2 Lw$$

$$= \left[0.073\left(\frac{5 \times 4}{1.6 \times 10^{-5}}\right)^{-1/5} - 1700\left(\frac{5 \times 4}{1.6 \times 10^{-5}}\right)^{-1}\right] \times \frac{1}{2} \times 1.16 \times 5^2 \times 4 \times 5$$

$$= \underline{0.88\text{ N}}$$

The prediction of part (a) is 45% too high. For relatively short surfaces it is obvious that significant errors result if the thinner laminar portion with its smaller shear stress is neglected.

8.6 Boundary-Layer Theory

8.6.5 Turbulent Boundary Layer: Empirical Form

- This method of understanding turbulent flow (flat plate, zero pressure gradient) uses obtained data.
  - More accurate that the power-law form but is harder to use.

- The time-average turbulent velocity profile can be divided into the *inner* and *outer* regions.
  - The inner region is defined as:

$$\frac{\bar{u}}{u_\tau} = f\left(\frac{u_\tau y}{\nu}\right) \qquad u_\tau = \sqrt{\frac{\tau_0}{\rho}}$$

u_T: shear velocity

  - The outer region is defined as:

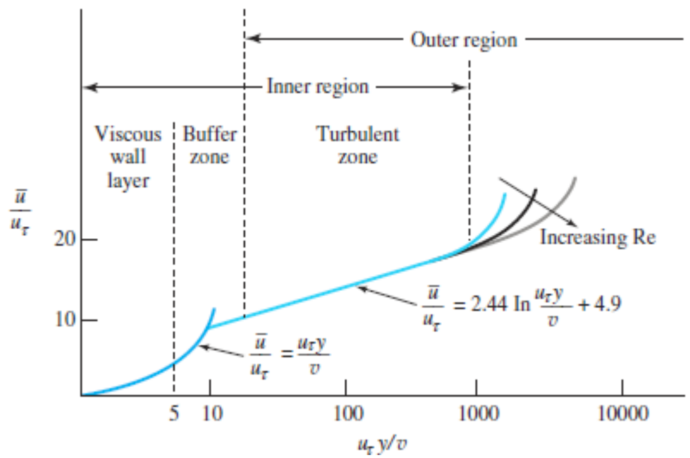
$$\frac{U_\infty - \bar{u}}{u_\tau} = f\left(\frac{y}{\delta}\right)$$

$U_\infty - \bar{u}$: Velocity defect

8.6 Boundary-Layer Theory

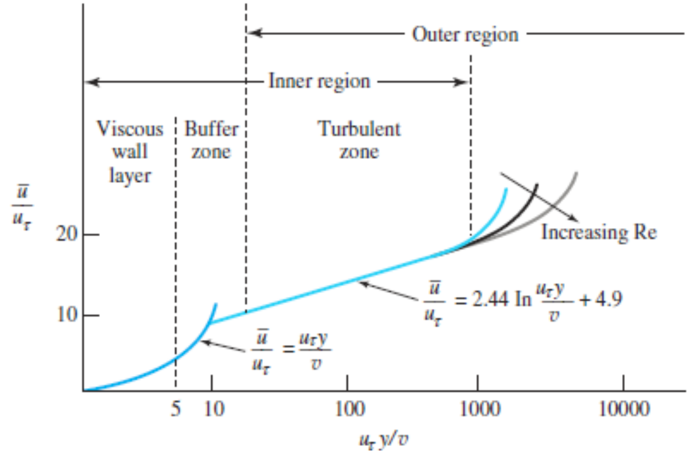
8.6.5 Turbulent Boundary Layer: Empirical Form

- The inner region has three regions: viscous wall layer, buffer zone, and turbulent zone.



8.6 Boundary-Layer Theory

8.6.5 Turbulent Boundary Layer: Empirical Form



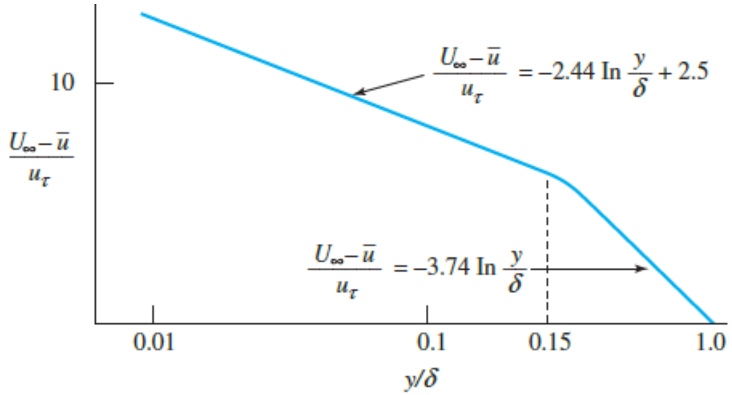
- The *viscous wall layer* (fluctuates constantly):
  - Defined as a linear time-average profile.
  - Very thin, extends to y* ≈ 5
  - v/u_T is the characteristic length in the turbulent inner region → The dimensionless distance from the wall is hence:

$$y^* = \frac{u_\tau y}{\nu}$$

- The *turbulent zone*:
  - Defined by a logarithmic profile.
  - From y* ≈ 50 to y/δ ≈ 0.15
  - Location of the outer edge depends on the Reynolds number
- The *buffer zone* connects the viscous wall layer and the turbulent zone.

8.6 Boundary-Layer Theory

8.6.5 Turbulent Boundary Layer: Empirical Form



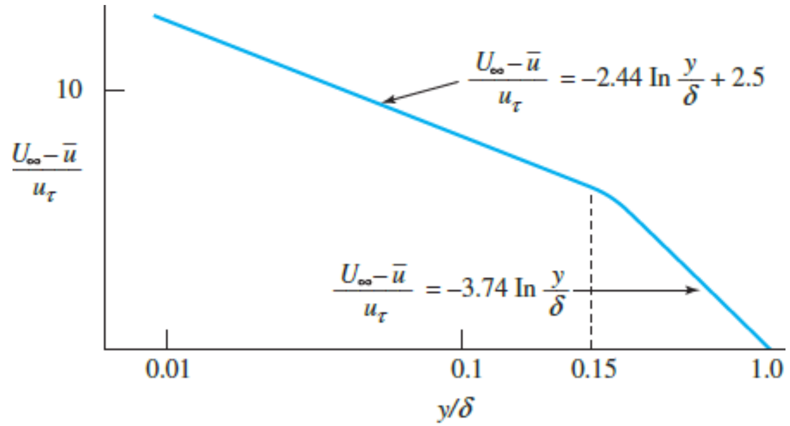
- The outer region relates the velocity defect to y/δ:
  - The turbulent zone is from $50 < \frac{u_\tau y}{\nu} \quad \frac{y}{\delta} < 0.15$
  - Above this range [$y/\delta > 0.15$] a data fit is used.

- These equations involve shear velocity u_T which depends on wall shear τ_0.
  - To find the wall shear (τ_0), the local skin-friction coefficient equation is needed as seen:

$$c_f = \frac{0.455}{(\ln 0.06\,\mathrm{Re}_x)^2}$$

8.6 Boundary-Layer Theory

8.6.5 Turbulent Boundary Layer: Empirical Form



- For turbulent flow from the leading edge, the shear stress can be integrated to find the drag.
  - The skin friction coefficient is then:

$$C_f = \frac{0.523}{(\ln 0.06 \mathrm{Re}_L)^2}$$

Accurate up to $Re_L = 10^9$

- For the common turbulent region, the two logarithmic profiles are combined as shown below:
  - Can easily find δ from u_T

$$\frac{U_\infty}{u_\tau} = 2.44 \ln \frac{u_\tau \delta}{\nu} + 7.4$$

8.6 Boundary-Layer Theory

Estimate the thickness δ_v of the viscous wall layer and the boundary-layer thickness at the end of a 4.5 m-long flat plate if $U_\infty = 30$ m/s in 20°C-atmospheric air. Also, calculate the drag force on one side if the plate is 3 m wide. Use the empirical data.

Solution

To find the viscous wall layer thickness we must know the shear velocity and hence the wall shear. The wall shear, using Eq. 8.6.40, and the shear velocity at $x = 4.5$ m are

$$\begin{aligned}\tau_0 &= \frac{1}{2}\rho U_\infty^2 c_f \\ &= \frac{1}{2}\rho U_\infty^2 \frac{0.455}{(\ln 0.06\,\mathrm{Re}_x)^2} \\ &= \frac{1}{2} \times 1.2\,\mathrm{kg/m^3} \times 30^2\,\mathrm{m^2/s^2} \frac{0.455}{\left(\ln 0.06 \dfrac{30 \times 4.5}{1.46 \times 10^{-5}}\right)^2} = 1.404\,\mathrm{Pa}\end{aligned}$$

$$u_\tau = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{1.404\,\mathrm{Pa}}{1.2\,\mathrm{kg/m^3}}} = 1.082\,\mathrm{m/s}$$

The viscous wall-layer thickness is determined using Eq. 8.6.36 with $y^* = 5$ as follows:

$$\frac{u_\tau \delta_v}{\nu} = 5$$

$$\delta_v = \frac{5\nu}{u_\tau} = \frac{5 \times 1.46 \times 10^{-5}}{1.082} = \underline{6.75 \times 10^{-5}\,\mathrm{m}}$$

8.6 Boundary-Layer Theory

The boundary-layer thickness is found using Eq. 8.6.42:

$$\frac{U_\infty}{u_\tau} = 2.44 \ln \frac{u_\tau \delta}{\nu} + 7.4$$

$$\frac{30}{1.082} = 2.44 \ln \frac{1.082 \times \delta}{1.46 \times 10^{-5}} + 7.4 \qquad \therefore\ \delta = \underline{0.056 \text{ m}}$$

The drag force is calculated using Eq. 8.6.41 to be

$$\begin{aligned} F_D &= C_f \times \frac{1}{2}\rho U_\infty L w \\ &= \frac{0.523}{(\ln 0.06 \mathrm{Re}_L)^2} \times \frac{1}{2}\rho U_\infty^2 L w \\ &= \frac{0.523}{\left(\ln 0.06 \dfrac{30 \times 4.5}{1.46 \times 10^{-5}}\right)^2} \times \frac{1}{2} \times 1.2 \text{ kg/m}^3 \times 30^2 \text{ m}^2\text{/s}^2\ (4.5 \times 3) \text{ m}^2 = \underline{21.8 \text{ N}} \end{aligned}$$

The laminar portion of the boundary layer has been neglected.

8.6 Boundary-Layer Theory

Estimate the maximum boundary-layer thickness and the drag due to friction on the side of a ship that measures 40 m long with a submerged depth of 8 m assuming the side of the ship is approximated as a flat plate. The ship travels at 10 m/s. (a) Use the empirical methods and (b) compare with the results using the power-law model.

Solution

(a) The boundary-layer thickness is found from Eq. 8.6.42. First we must find τ_0 from Eq. 8.6.40 and then u_τ as follows:

$$\tau_0 = \frac{1}{2}\rho U_\infty^2 \frac{0.455}{(\ln 0.06\,\mathrm{Re}_L)^2}$$

$$= \frac{1}{2} \times 1000\,\mathrm{kg/m^3} \times 10^2\,\mathrm{m^2/s^2} \frac{0.455}{\left(\ln 0.06 \dfrac{10 \times 40}{10^{-6}}\right)^2} = 78.8\,\mathrm{Pa}$$

$$\therefore u_\tau = \sqrt{\frac{\tau_0}{\rho}}$$

$$= \sqrt{\frac{78.8\,\mathrm{N/m^2}}{1000\,\mathrm{kg/m^3}}} = 0.28\,\mathrm{m/s}$$

The maximum boundary-layer thickness is found using Eq. 8.6.42:

$$\frac{U_\infty}{u_\tau} = 2.44 \ln \frac{u_\tau \delta}{\nu} + 7.4$$

$$\frac{10}{0.28} = 2.44 \ln \frac{0.28\,\delta}{10^{-6}} + 7.4 \qquad \therefore \delta = \underline{0.39\ \mathrm{m}}$$

8.6 Boundary-Layer Theory

The drag is

$$F_D = C_f \times \tfrac{1}{2}\rho U_\infty^2 Lw$$
$$= \frac{0.523}{\left(\ln 0.06 \dfrac{10 \times 40}{10^{-6}}\right)^2} \times \frac{1}{2} \times 1000 \times 10^2 \times 40 \times 8 = \underline{29000\text{ N}}$$

(b) First, calculate the Reynolds number: $\text{Re} = 10 \times 40/10^{-6} = 4 \times 10^8$. We select $n = 9$. Equation (8.6.25) becomes

$$\tau_0 = \frac{d}{dx}\int_0^\delta \rho U_\infty^2 \left(\frac{y}{\delta}\right)^{1/9}\left[1 - \left(\frac{y}{\delta}\right)^{1/9}\right] dy$$
$$= \frac{9}{110}\rho U_\infty^2 \frac{d\delta}{dx}$$

Equating this to the τ_0 of Eq. 8.6.24, we find that

$$\delta^{1/4}\, d\delta = 0.281(\nu/U_\infty)^{1/4}\, dx$$

Assume $\delta = 0$ at $x = 0$ and integrate. This provides

$$\delta = 0.433x\,\text{Re}_x^{-1/5}$$
$$= 0.433(40)\left(\frac{10 \times 40}{10^{-6}}\right)^{-1/5} = \underline{0.33\text{ m}}$$

This value is 15% too low.

The drag force is found to be

$$F_D = 0.071\text{Re}_L^{-1/5} \times \tfrac{1}{2}\rho U_\infty^2 Lw$$
$$= 0.071\left(\frac{10 \times 40}{10^{-6}}\right)^{-1/5} \times \frac{1}{2} \times 1000 \times 10^2 \times 40 \times 8 = \underline{21600\text{ N}}$$

This value is 25% too low. Obviously, the power-law equations are in significant error.

8.6 Boundary-Layer Theory

8.6.6 Laminar Boundary-Layer Equations

- The solution presented in Section 8.6.3 for the laminar boundary layer was an approximate solution using a cubic polynomial to approximate the velocity profile.
  - The Navier-Stokes equations can be simplified to find a better solution.

- Assuming steady, incompressible, plane flow, the Navier-Stokes equation and continuity equation become:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \longrightarrow \boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}}$$

Prandtl Boundary-Layer Equation

8.6 Boundary-Layer Theory

8.6.6 Laminar Boundary-Layer Equations

- The solution for the laminar boundary layer with dp/dx = 0 is:

$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$	F	$F' = u/U_\infty$	$\frac{1}{2}(\eta F' - F)$	F''
0	0	0	0	0.3321
1	0.1656	0.3298	0.0821	0.3230
2	0.6500	0.6298	0.3005	0.2668
3	1.397	0.8461	0.5708	0.1614
4	2.306	0.9555	0.7581	0.0642
5	3.283	0.9916	0.8379	0.0159
6	4.280	0.9990	0.8572	0.0024
7	5.279	0.9999	0.8604	0.0002
8	6.279	1.0000	0.8605	0.0000

- With: $\delta = 5\sqrt{\frac{\nu x}{U_\infty}}$ and: $\tau_0 = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0.332\rho U_\infty^2 \sqrt{\frac{\nu}{x U_\infty}}$

8.6 Boundary-Layer Theory

8.6.6 Laminar Boundary-Layer Equations

- The local and general skin friction coefficients are then found to be:

$$c_f = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

$$C_f = \frac{1.33}{\sqrt{\mathrm{Re}_L}}$$

- The displacement and momentum thicknesses are:

$$\delta_d = 1.72\sqrt{\frac{\nu x}{U_\infty}} \qquad \theta = 0.644\sqrt{\frac{\nu x}{U_\infty}}$$

8.6 Boundary-Layer Theory

Atmospheric air at 30°C flows over a 8-m-long, 2-m-wide flat plate at 2 m/s. Assume that laminar flow exists in the boundary layer over the entire length. At $x = 8$ m, calculate (a) the maximum value of v, (b) the wall shear, and (c) the flow rate through the layer. (d) Also, calculate the drag force on the plate.

Solution

(a) The y-component of velocity has been assumed to be small in boundary-layer theory. Its maximum value at $x = 8$ m is found, using 8.6.51, to be

$$
\begin{aligned}
v &= \sqrt{\frac{\nu U_\infty}{x}} \times \frac{1}{2}(\eta F' - F) \\
&= \sqrt{\frac{1.6 \times 10^{-5} \times 2}{8}} \times 0.86 = \underline{0.00172 \text{ m/s}}
\end{aligned}
$$

where 0.86 comes from Table 8.5. Compare v with $U_\infty = 2$ m/s.

(b) The wall shear at $x = 8$ m is found using Eq. 8.6.56 to be

$$
\begin{aligned}
\tau_0 &= 0.332 \rho U_\infty^2 \sqrt{\frac{\nu}{U_\infty x}} \\
&= 0.332 \times 1.16 \text{ kg/m}^3 \times 2^2 \text{ m}^2/\text{s}^2 \sqrt{\frac{1.6 \times 10^{-5} \text{ m}^2/\text{s}}{2 \text{ m}^2/\text{s} \times 8 \text{ m}}} \\
&= \underline{0.00154 \text{ Pa}}
\end{aligned}
$$

8.6 Boundary-Layer Theory

(c) The flow rate through the boundary layer at $x = 8$ m is given by

$$Q = \int_0^{\delta} u \times w dy = w\sqrt{\frac{\nu x}{U_\infty}} \int_0^{5} U_\infty F' \, d\eta$$

where we have substituted for u and y from Eqs. 8.6.51 and 8.6.48. Recognizing that $\int F' \, d\eta = F$, the flow rate is

$$Q = wU_\infty \sqrt{\frac{\nu x}{U_\infty}} [F(5) - \cancelto{0}{F(0)}]$$

$$= 2\text{m} \times 2\,\text{m/s} \sqrt{\frac{1.6 \times 10^{-5}\,\text{m}^2/\text{s} \times 8\text{m}}{2\,\text{m/s}}} \times 3.28 = \underline{0.105\text{m}^3\cdot\text{s}}$$

(d) The drag force is determined to be

$$F_D = \frac{1}{2}\rho U_\infty^2 L w C_f$$

$$= \frac{1}{2} \times 1.16\,\text{kg/m}^3 \times 2^2\,\text{m}^2/\text{s}^2 \times 8\,\text{m} \times 2\,\text{m} \times \frac{1.33}{\sqrt{2 \times 8/1.6 \times 10^{-5}}}$$

$$= \underline{0.049\text{ N}}$$

8.6 Boundary-Layer Theory

8.6.7 Pressure-Gradient Effects

- If a pressure gradient is applied, the boundary-layer flow is affected.
  - A large, negative pressure gradient can relaminarize a turbulent boundary layer.
  - A positive pressure gradient causes the boundary layer to thicken and separate.

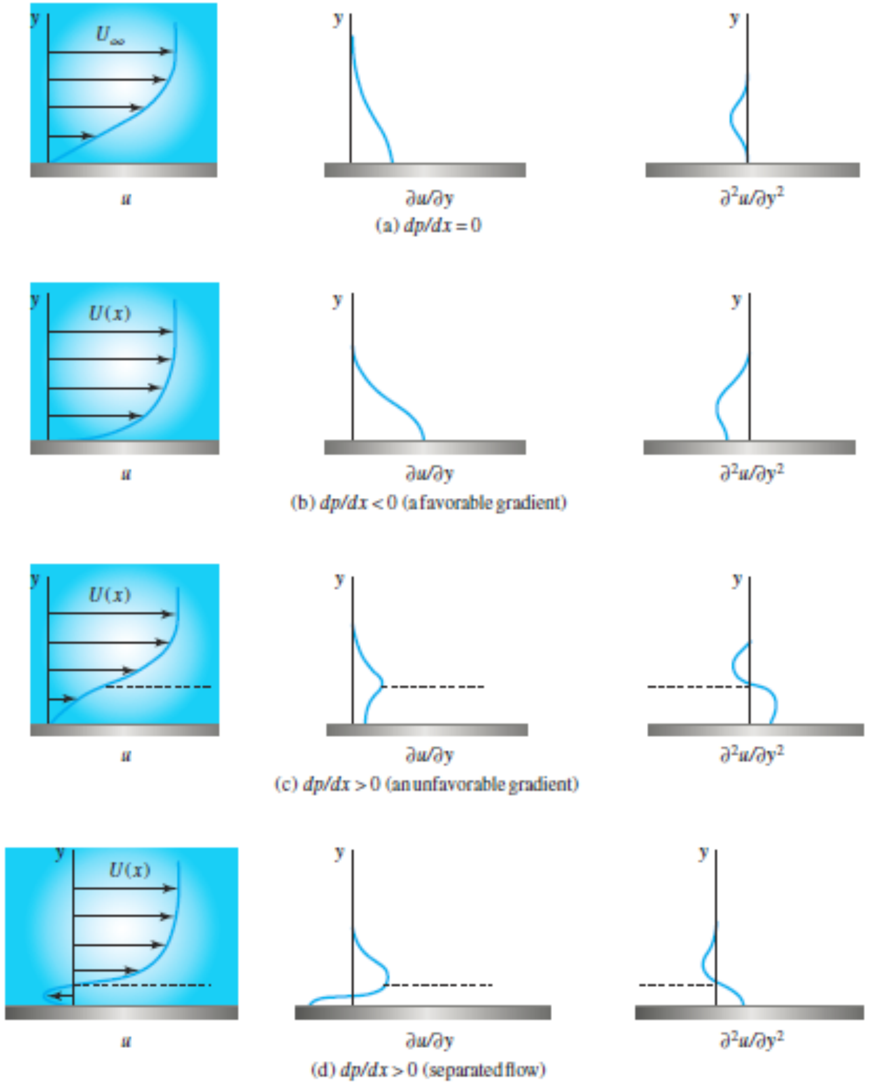


Figure 8.28 Influence of the pressure gradient.

8.7 Summary

- Drag and Lift coefficients are: $C_D = \dfrac{\text{Drag}}{\frac{1}{2}\rho V^2 A}$ $C_L = \dfrac{\text{Lift}}{\frac{1}{2}\rho V^2 A}$

- Vortex shedding occurs from a cylinder when 300 < Re < 10,000
  - The frequency of shedding is found from the Strouhal number. $\text{St} = \dfrac{fD}{V}$

- Plane potential flows can be found by superimposing simple flows below.

$$
\begin{array}{lll}
\text{Uniform flow:} & \psi = U_\infty y & \phi = U_\infty x \\
\text{Line source:} & \psi = \dfrac{q}{2\pi\theta} & \phi = \dfrac{q}{2\pi}\ln r \\
\text{Irrotational vortex:} & \psi = \dfrac{\Gamma}{2\pi}\ln r & \phi = \dfrac{\Gamma}{2\pi}\theta \\
\text{Doublet:} & \psi = -\dfrac{\mu}{r}\sin\theta & \phi = -\dfrac{\mu}{r}\cos\theta
\end{array}
$$

- The stream function for a rotating cylinder is: $\psi_{\text{cylinder}} = U_\infty y - \dfrac{\mu}{r}\sin\theta + \dfrac{\Gamma}{2\pi}\ln r$
  - With the cylinder radius: $r_c = \sqrt{\dfrac{\mu}{U_\infty}}$

8.7 Summary

- The velocity components are:

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

- For a laminar boundary layer on a flat plate (zero pressure gradient), the exact solution is:

$$\delta = 5\sqrt{\frac{\nu x}{U_\infty}} \qquad c_f = 0.664\sqrt{\frac{\nu}{xU_\infty}} \qquad C_f = 1.33\sqrt{\frac{\nu}{LU_\infty}}$$

- For a turbulent flow, the power-law profile (η = 7):

$$\delta = 0.38x\left(\frac{\nu}{xU_\infty}\right)^{1/5} \qquad c_f = 0.059\left(\frac{xU_\infty}{\nu}\right)^{1/5} \qquad C_f = 0.073\left(\frac{\nu}{LU_\infty}\right)^{1/5}$$

  - The wall shear and drag force per unit width are:

$$\tau_0 = \frac{1}{2}c_f \rho U_\infty^2 \qquad F_D = \frac{1}{2}C_f \rho U_\infty^2 L$$